Representation Theory: Making abstract algebra more concrete

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What is representation theory?

- Consider the following matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- How would we study/understand it?
  - How about how it acts on the vector spaces $\mathbb{R}^4$ or $\mathbb{C}^4$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ x_3 \\ x_4 \end{pmatrix}$$

- Something we see in abstract algebra is S_n, the *symmetric group on n elements*, which is made up of permutations acting the set $\{1, 2, 3, \ldots, n\}$.

Example 1

Consider $(12) \in S_4$. It switches the position of 1 and 2 in the set, that is,

$$(12).\{1, 2, 3, 4\} \to \{2, 1, 3, 4\}$$

- How we study this is built into its definition. Understanding how it acts on a set.

- This is a theme of representation theory. Understanding an object by how it acts on something else.
- In particular, we take objects from (very) abstract algebra and relate them to linear algebra.
- Things like:
  - The symmetric group [Jam84],
  - Lie algebras [Hum78],and
  - Galois and flag orders [FO10; Har20; Web19; Jau21; Jau22]

Example - Symmetric group

Example 2

We can take $S_4 \to \{4x4 \text{ Permutation Matrices}\}$ for example we have

$$(12) \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This allows S_4 to act on $\mathbb{C}^4$. Where it takes

$$(12).(x_1, x_2, x_3, x_4)^T \to (x_2, x_1, x_3, x_4)^T$$

Example - Lie algebras

Example 3

- We call $GL_n(\mathbb{C})$ the *the general linear group of $n \times n$ matrices (over $\mathbb{C}$)* which is the collection of every invertible $n \times n$ matrix.
- To understand that we actually relate it to $\mathfrak{gl}_n(\mathbb{C})$ the *general linear algebra of $n \times n$ matrices (over $\mathbb{C}$)* which is the collection of EVERY $n \times n$ matrix.
- To understand THAT we relate it to $U(\mathfrak{gl}_n)$ the *universal enveloping algebra of* $\mathfrak{gl}_n(\mathbb{C})$ which is complicated...

Smash product

- We need two things to build the algebras I care about.
  - Λ - a Noetherian integrally closed domain (and Frac(Λ) its field of fractions)
  - $\hat{W}$ - a subgroup of Aut(Λ)

Definition 4

The *smash product* $\text{Frac}(\Lambda)\#\hat{W}$ is the collection of formal sums:

$$X = \sum_{w \in \hat{W}} f_w w$$

where $f_w \in \text{Frac}(\Lambda)$ is non-zero for finitely many f_w.

- We add terms together component-wise.
- Multiplication is more interesting:

$$f_1 w_1 \cdot f_2 w_2 = (f_1 w_1(f_2))(w_1 w_2)$$

Example

- Let's look at something concrete to hold onto.

Example 5

Let $\Lambda = \mathbb{C}[x_1, x_2]$, complex polynomials in two variables, and $\hat{W} = S_2$, the permutation group. Then

$$\mathbb{C}(x_1, x_2)\#S_2 = \left\{ \sum_{\sigma \in S_2} f_\sigma \sigma \;\middle|\; f_\sigma \text{ a rational function} \right\}.$$

Let's compute $(x_1 - x_2)(12) \cdot (x_1 - x_2)$

$$\begin{aligned}(x_1 - x_2)(12) \cdot (x_1 - x_2) &= (x_1 - x_2)(12)((x_1 - x_2))(12) \\ &= (x_1 - x_2)(x_2 - x_1)(12) \\ &= -(x_1 - x_2)^2(12)\end{aligned}$$

Standard flag order

- Since the terms we are summing over are maps, the elements in the smash product can be thought of as maps.
- If $X = \sum_{w \in \hat{W}} f_w w$, then for $g \in \mathrm{Frac}(\Lambda)$,

$$X(g) = \sum_{w \in \hat{W}} f_w w(g)$$

Definition 6 (from [Web19])

We define the *standard flag order* as all the $X \in \mathrm{Frac}(\Lambda) \# \hat{W}$ such that $X(\Lambda) \subset \Lambda$. We denote it by $\mathcal{F}_\Lambda$.

Example - how the maps work

- We'll look at something in the standard flag order in $\mathbb{C}(x_1, x_2)\#S_2$

Example 7

Consider the following element:

$$\sigma_1 := \frac{1}{x_2 - x_1}(12) - \frac{1}{x_2 - x_1}$$

We consider $\sigma_1(g(x_1, x_2))$:

$$\begin{aligned}\sigma_1(g(x_1, x_2) &= \frac{1}{x_2 - x_1}(12)(g(x_1, x_2)) - \frac{g(x_1, x_2)}{x_2 - x_1} \\ &= \frac{g(x_2, x_1)}{x_2 - x_1} - \frac{g(x_1, x_2)}{x_2 - x_1} \\ &= \frac{g(x_2, x_1) - g(x_1, x_2)}{x_2 - x_1} \in \mathbb{C}[x_1, x_2]\end{aligned}$$

Examples - standard flag orders

Example 8

If $\Lambda = \mathbb{C}[x_1, x_2, \ldots, x_n]$ and $\hat{W} = S_n$, then the standard flag order is

$$\mathcal{F}_\Lambda = \Lambda[\sigma_1, \ldots, \sigma_{n-1}] \text{ where } \sigma_i = \frac{1}{x_{i+1} - x_i}(i, i+1) - \frac{1}{x_{i+1} - x_i}$$

Example 9

If $\Lambda = \mathbb{C}[x]$ and $\hat{W} = \langle \tau \rangle$ where $\tau(f(x)) = f(-x)$., then the standard flag order is

$$\mathcal{F}_\Lambda = \Lambda\left[\frac{1}{x}\tau - \frac{1}{x}\right]$$

Why?

- Linear algebra is a rigid and well-understood field, whereas abstract algebra is abstract.
- It can link several seemingly distinct objects together by having similar representations.
- Flag orders unify the representation theory of many important objects in algebra.
- Connections to mathematical physics and category theory.

What do I do?

- In algebra one of the first steps after defining an object is to describe maps between those objects.
- In my pre-print [Jau22] I have described sufficient conditions for maps between not only flag orders, but also Galois orders (see [FO10] and [Har20]).
- Additionally, I showed that flag (and Galois) orders are closed under tensor products thereby taking the first steps toward describing the category of flag (and Galois) orders.

References

[FO10] Vyacheslav Futorny and Serge Ovsienko. "Galois orders in skew monoid rings". In: *Journal of Algebra* 324.4 (2010), pp. 598–630. ISSN: 0021-8693. DOI: https://doi.org/10.1016/j.jalgebra.2010.05.006. URL: http://www.sciencedirect.com/science/article/pii/S0021869310002267.

[Har20] Jonas T. Hartwig. "Principal Galois orders and Gelfand-Zeitlin modules". In: *Advances in Mathematics* 359 (2020), p. 106806. ISSN: 0001-8708. DOI: https://doi.org/10.1016/j.aim.2019.106806. URL: http://www.sciencedirect.com/science/article/pii/S0001870819304232.

[Hum78] James E. Humphreys. *Introduction to Lie algebras and representation theory*. Vol. 9. Graduate Texts in Mathematics. Second printing, revised. Springer-Verlag, New York-Berlin, 1978, pp. xii+171. ISBN: 0-387-90053-5.

[Jam84] Gordon James. *The Representation Theory of the Symmetric Group*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1984. DOI: 10.1017/CBO9781107340732.

[Jau21] Erich C. Jauch. "An extension of $U(\mathfrak{gl}_n)$ related to the alternating group and Galois orders". In: *Journal of Algebra* 569 (2021), pp. 568–594. ISSN: 0021-8693. DOI: https://doi.org/10.1016/j.jalgebra.2020.10.017. URL: http://www.sciencedirect.com/science/article/pii/S0021869320305469.

[Jau22] Erich C. Jauch. *Maps between standard and principal flag orders*. 2022. DOI: 10.48550/ARXIV.2208.13117. URL: https://arxiv.org/abs/2208.13117.

[Web19] Ben Webster. "Gelfand-Tsetlin modules in the Coulomb context". In: *arXiv e-prints* (Apr. 2019). arXiv: 1904.05415 [math.RT].

Thank you. Questions?