

Galois Orders

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History and Motivation

- Studying “subalgebra-algebra” pairs finds its roots in representation theory. We will be looking at “semicommutative” pairs $\Gamma \subset \mathcal{U}$ where \mathcal{U} is an associative (noncommutative) \mathbb{C} -algebra and Γ an integral domain.
- Motivation for such pairs comes from framework of *Harish-Chandra modules* where \mathcal{U} is the universal enveloping algebra of a reductive Lie algebra and Γ is the universal enveloping algebra of a Cartan subalgebra (generalized weight modules) [DFO2].
- The objects described here were originally defined and studied by Futorny and Ovsienko in [FO10] and [FO14].
- *Galois rings* and *Galois orders* form a collection of algebras that contains many important examples:
 - *Generalized Weyl algebras* (Bavula, Rosenberg '9*)
 - Universal enveloping algebra of \mathfrak{gl}_n
 - Shifted Yangians and Finite W -algebras

Definitions

- We will follow the setting of Hartwig's from [Har17].
- Let Λ be a Noetherian closed domain, G a subgroup of $\text{Aut}(\Lambda)$, and \mathcal{M} a separating submonoid of $\text{Aut}(\Lambda)$ with respect to G such that G acts by conjugation on it.
- Let $\Gamma := \Lambda^G$

Definition 1

Given a commutative ring R and a submonoid $\mathcal{M} \subseteq \text{Aut}(R)$, we define the *smash product* as follows:

$$R\#\mathcal{M} := \left\{ \sum_{\mu \in \mathcal{M}} a_{\mu}\mu \mid a_{\mu} \in R \text{ and finitely many } a_{\mu} \neq 0 \right\},$$

with component-wise addition, and multiplication defined by $a_1\mu_1 \cdot a_2\mu_2 = (a_1\mu_1(a_2))\mu_1\mu_2$ and expanding linearly.

- Since G acts on Λ , its action naturally extends to an action on $\text{Frac}(\Lambda)$.
- As such, G acts on $\text{Frac}(\Lambda)\#\mathcal{M}$.
- We have the following diagram:

$$\begin{array}{ccccc}
 \Lambda & \hookrightarrow & \text{Frac}(\Lambda) & \hookrightarrow & \text{Frac}(\Lambda)\#\mathcal{M} \\
 \uparrow & & \uparrow & & \uparrow \\
 \Gamma & \hookrightarrow & \text{Frac}(\Gamma) & \hookrightarrow & (\text{Frac}(\Lambda)\#\mathcal{M})^G
 \end{array}$$

- Note: $\text{Frac}(\Lambda)/\text{Frac}(\Gamma)$ is a Galois extension with Galois group G .

Definition 2

For an element $X \in (\text{Frac}(\Lambda)\#\mathcal{M})^G$ of the form $X = \sum a_\mu \mu$, we define $\text{supp}_{\mathcal{M}}(X) = \{\mu \mid a_\mu \neq 0\}$.

Definition 3 (Futorny-Ovsienko 2010)

A *Galois Γ -ring* is a subalgebra \mathcal{U} of $(\text{Frac}(\Lambda)\#\mathcal{M})^G$ containing Γ such that $\text{Frac}(\Gamma)\mathcal{U} = \mathcal{U} \text{Frac}(\Gamma) = (\text{Frac}(\Lambda)\#\mathcal{M})^G$.

We have the following criterion for Galois rings:

Proposition 4 (Furtony-Ovsienko 2010)

Let $\mathcal{X} \subseteq (\text{Frac}(\Lambda)\#\mathcal{M})^G$ and let \mathcal{U} the the subring of $(\text{Frac}(\Lambda)\#\mathcal{M})^G$ generated by $\Gamma \cup \mathcal{X}$. Then \mathcal{U} is a Galois Γ -ring iff $\cup_{X \in \mathcal{X}} \text{supp}_{\mathcal{M}}(X)$ generates \mathcal{M} as a monoid.

Definition 5 (Futorny-Ovsienko 2010)

A *Galois Γ -order* is a Galois Γ -ring such that for any finite dimensional left (or right) $\text{Frac}(\Gamma)$ -subspace W of $(\text{Frac}(\Lambda) \# \mathcal{M})^G$, $\mathcal{U} \cap W$ is a finite generated left (resp. right) Γ -module.

- The above condition is very technical and difficult to show.
- In 2017, Hartwig showed the following condition implies the above condition:

Definition 6 (Hartwig 2017)

Let \mathcal{U} be a Galois Γ -ring such that $X(\Gamma) \subseteq \Gamma$ for every $X \in \mathcal{U}$. Then \mathcal{U} is a principal Galois Γ -order

- Note: Γ is maximal commutative in any Galois Γ -order.

Definition 7

A \mathcal{U} -module V is a *Gelfand-Zeitlin* module (with respect to Γ) if $\dim(\Gamma.v) < \infty$ for all $v \in V$.

The major results in [FO14] give:

- 1 The existence of “generic” simple Gelfand-Zeitlin modules over Galois rings.
- 2 A “rough” classification of simple Gelfand-Zeitlin modules over Galois orders.

A Simple Example

- Let $\Lambda = \mathbb{C}[x]$, $\delta \in \text{Aut}(\Lambda)$ such that $\delta(x) = x - 1$, $\mathcal{M} = \langle \delta \rangle_{\text{grp}} \cong \mathbb{Z}$, and G the trivial group.
- $f(x) \in \mathbb{C}[x]$ such that $f(0) \neq 0$
- Define $X, Y \in \text{Frac}(\Lambda) \# \mathcal{M}$ such that

$$X := \delta \frac{f(x)}{x} \quad \text{and} \quad Y := \delta^{-1}.$$

- Let $\mathcal{U}_f = \mathbb{C}\langle \Lambda, X, Y \rangle_{\text{alg}}$.
- Then \mathcal{U}_f is a Galois Λ -ring by the Galois ring criterion because $\text{supp}_{\mathcal{M}} X \cup \text{supp}_{\mathcal{M}} Y = \{\delta, \delta^{-1}\}$ which generate \mathcal{M} .

Is \mathcal{U}_f a Galois Λ -order?

- Let $C := C_{\mathcal{U}_f}(\Lambda) = \{u \in \mathcal{U}_f \mid fu = uf \ \forall f \in \Lambda\}$, the centralizer of Λ in \mathcal{U}_f .
- Since $YX = \frac{f(x)}{x} \in C \setminus \Lambda$, Λ is not maximal commutative in \mathcal{U}_f .
- Since Λ is not maximal commutative, \mathcal{U}_f is not a Galois Λ -order.
- However, C is maximal commutative.

Question 2.1

Is \mathcal{U}_f a Galois C -order?

- We first need to describe C .

Describing \mathcal{C}

Lemma 8

For any $f(x)$ such that $f(0) \neq 0$, $\frac{1}{x}, \frac{1}{x-1} \in \mathcal{C}$.

Lemma 9

For any $f(x)$ such that $f(0) \neq 0$ and $k \geq 1$, $\frac{1}{x+k} \in \mathcal{C}$.

Lemma 10

For any $f(x)$ such that $f(0) \neq 0$ and $k \geq 2$, $\frac{1}{x-k} \in \mathcal{C}$.

Proposition 11 (J* 2019)

If $f(x)$ is a polynomial such that $f(0) \neq 0$, then

$$C = \mathbb{C}[x] \left[\frac{1}{x+k} \mid k \in \mathbb{Z} \right].$$

Theorem 12 (J* 2019)

If $f(x)$ is a polynomial such that $f(0) \neq 0$, then \mathcal{U}_f is a (co-)principal Galois C -order.

Set-Up

- We recall that $U(\mathfrak{gl}_n)$ -modules can be represented by Gelfand-Tsetlin patterns.

Example 13

Let $(\lambda_{21}, \lambda_{22})$ be a $U(\mathfrak{gl}_2)$ weight. Then the following is a Gelfand-Tsetlin pattern $L(\lambda_{21}, \lambda_{22})$:

$$\begin{array}{cc} \lambda_{21} & \lambda_{22} \\ & \lambda_{11} \end{array}$$

where $\lambda_{21} \geq \lambda_{11} \geq \lambda_{22}$.

- $U(\mathfrak{gl}_2)$ acts on these patterns via rational functions in the entries λ_{ij} and integral shifts.

- In 2010, Futorny and Ovsienko gave a realization of $U(\mathfrak{gl}_n)$ as a Galois order with $\Lambda = \mathbb{C}[x_{ki} \mid 1 \leq i \leq k \leq n]$, $G = S_1 \times S_2 \times \cdots \times S_n$, and $\mathcal{M} = \langle \delta^{j\ell} \mid 1 \leq \ell \leq j \leq n-1 \rangle_{\text{grp}}$ where $\delta^{j\ell}(x_{ki}) = x_{ki} - \delta_{jk}\delta_{i\ell}$.
- We will use U_n to denote the image of $U(\mathfrak{gl}_n)$ under the embedding $\varphi: U(\mathfrak{gl}_n) \hookrightarrow (\text{Frac}(\Lambda) \# \mathcal{M})^G$ defined as follows:

$$\varphi(E_k^\pm) = \sum_{i=1}^k (\delta^{ki})^{\pm 1} a_{ki}^\pm \quad \text{with} \quad a_{ki}^\pm = \mp \frac{\prod_{j=i}^{k\pm 1} x_{k\pm 1, j} - x_{ki}}{\prod_{j \neq i} x_{kj} - x_{ki}},$$

$$\varphi(E_{kk}) = \sum_{i=1}^k (x_{ki} + i - 1) - \sum_{i=1}^{k-1} (x_{(k-1)i} + i - 1).$$

Example 14

For $n = 2$, U_2 is contained in

$$(\mathbb{C}(x_{11}, x_{21}, x_{22}) \# \langle \delta^{11} \rangle_{\text{grp}})^{S_1 \times S_2}$$

where δ^{11} is an automorphism of $\mathbb{C}(x_{11}, x_{21}, x_{22})$ defined by

$$\delta^{11}(x_{ij}) = x_{ij} - \delta_{1i}\delta_{1j}.$$

with generators:

$$\varphi(E_1^+) = -\delta^{11}(x_{21} - x_{11})(x_{22} - x_{11}),$$

$$\varphi(E_1^-) = (\delta^{11})^{-1},$$

$$\varphi(E_{11}) = x_{11},$$

$$\varphi(E_{22}) = x_{21} + x_{22} - x_{11} + 1.$$

- What happens when we change the **symmetric groups** to **alternating groups**?
- Recall that $\mathbb{C}[x_1, x_2, \dots, x_n]^{A_n} = \mathbb{C}[x_1, x_2, \dots, x_n]^{S_n}[\mathcal{V}]$.

Definition 15 (J* 2019)

The *alternating analogue* of $U(\mathfrak{gl}_n)$, denoted $\mathcal{A}(\mathfrak{gl}_n)$, is defined as the subalgebra of $(\text{Frac}(\Lambda) \# \mathcal{M})^{A_1 \times A_2 \times \dots \times A_n}$ generated by $U_n \cup \{\mathcal{V}_2, \mathcal{V}_3, \dots, \mathcal{V}_n\}$ where:

$$\mathcal{V}_k = \mathcal{V}_k(x_{k1}, \dots, x_{kk}) = \prod_{i < j} (x_{ki} - x_{kj}) \quad \text{for } k = 1, \dots, n-1.$$

- The following proposition lists some basic properties of $\mathcal{A}(\mathfrak{gl}_n)$.

Proposition 16 (J* 2019)

- (i) $U(\mathfrak{gl}_n) \cong U_n \subset \mathcal{A}(\mathfrak{gl}_n)$,
- (ii) $\mathcal{A}(\mathfrak{gl}_n)$ is a Galois ring,
- (iii) \mathcal{V}_n is central in $\mathcal{A}(\mathfrak{gl}_n)$,
- (iv) $Z(\mathcal{A}(\mathfrak{gl}_n)) \cong \mathbb{C}[x_1, \dots, x_n]^{A_n}$,
- (v) there is a chain of subalgebras $\mathcal{A}(\mathfrak{gl}_1) \subset \mathcal{A}(\mathfrak{gl}_2) \subset \dots \subset \mathcal{A}(\mathfrak{gl}_n)$,
- (vi) $\mathcal{A}(\mathfrak{gl}_n)$ is the minimal extension of $U(\mathfrak{gl}_n)$ with properties (iv) and (v).

The case $n = 2$

Proposition 17 (J* 2019)

$\mathcal{A}(\mathfrak{gl}_2)$ is isomorphic to the following extension of $U(\mathfrak{gl}_2)$:

$$\frac{U(\mathfrak{gl}_2)[T]}{(T^2 - (-c_{21}^2 + 2c_{22} + 1))}$$

where c_{2i} are the so-called Gelfand invariants for \mathfrak{gl}_2 with

$$c_{21} = E_{11} + E_{22} \quad \text{and} \quad c_{22} = E_{11}^2 + E_{22}^2 + E_{21}E_{12} + E_{12}E_{21}.$$

Theorem 18 (J*, 2019)

$\mathcal{A}(\mathfrak{gl}_2)$ is a Galois order.

Proof idea.

We use the previous proposition, \mathcal{V}_2 is central, and a theorem of Futorny and Ovsienko from [FO10].

Larger n

- Unfortunately the following example shows that $\mathcal{A}(\mathfrak{gl}_n)$ is not a Galois order for $n \geq 3$.

Example 19

Let $[\cdot, \cdot]$ denote the commutator bracket. Then

$$\left(\varphi(E_2^+) + [\varphi(E_2^+), \nu_2] \right) \cdot \left(\varphi(E_2^-) - [\varphi(E_2^-), \nu_2] \right) = \delta^{21} a_{21}^+ \cdot (\delta^{21})^{-1} a_{21}^-,$$

an element centralizer of Γ in $\mathcal{A}(\mathfrak{gl}_n)$.

For $n = 2$

- The structure of $\mathcal{A}(\mathfrak{gl}_n)$ -modules is related to $U(\mathfrak{gl}_n)$ -modules in an interesting way.

Proposition 20 (J* 2019)

The finite-dimensional simple $\mathcal{A}(\mathfrak{gl}_2)$ -modules are characterized by ordered pairs $(\lambda_2, \varepsilon_2)$, where $\lambda_2 := (\lambda_{21}, \lambda_{22}) \in \mathbb{C}^2$ is a dominant integral weight for $U(\mathfrak{gl}_2)$ and $\varepsilon_2 \in \{1, -1\}$.

- We note that \mathcal{V}_2^2 must act diagonally on any finite-dimensional $\mathcal{A}(\mathfrak{gl}_2)$ -module V .
- $\text{Res}_{U(\mathfrak{gl}_2)}^{\mathcal{A}(\mathfrak{gl}_2)} V$ is a direct sum of simple $U(\mathfrak{gl}_2)$ -modules and \mathcal{V}_2^2 is a quadratic polynomial of Gelfand invariants in $U(\mathfrak{gl}_2)$.

Example 21

Let $V = V(0) \oplus V(0)$ where $U(\mathfrak{gl}_2)$ acts trivially. This means that \mathcal{V}_2^2 must act as Id_V . We define the following action of \mathcal{V}_2

$$\mathcal{V}_2 \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

with $0 \neq \alpha \in \mathbb{C}$. It is clear then that \mathcal{V}_2^2 acts as the identity on V , but the subrepresentation $W = \{(v_1, 0) \mid v_1 \in V(0)\}$ is not a direct summand of V as a $\mathcal{A}(\mathfrak{gl}_2)$ -module.

Larger n

Theorem 22 (J* 2019)

Every finite-dimensional simple module over $\mathcal{A}(\mathfrak{gl}_n)$, on which $\mathcal{V}_2, \dots, \mathcal{V}_{n-1}$ act diagonally, is of the form $V(\lambda_n, \varepsilon_n, \varepsilon_{n-1}, \dots, \varepsilon_2)$ where $\lambda_n = (\lambda_{n1}, \lambda_{n2}, \dots, \lambda_{nn})$ is a weight of $U(\mathfrak{gl}_n)$, $\varepsilon_j \in \{\pm 1\}^{r_{\lambda_n, j}}$, with $r_{\lambda_n, j}$ denoting the number of ways to fill the j -th row of Gelfand-Tsetlin pattern with fixed top row λ_n , and $j = 2, 3, \dots, n$.

Proof idea.

Follows by induction on n , and the following commutative diagram:

$$\begin{array}{ccccccc}
 \mathcal{A}(\mathfrak{gl}_n) - \underline{\text{Mod}}^{\text{f.d.}} & \longrightarrow & \mathcal{A}(\mathfrak{gl}_{n-1}) - \underline{\text{Mod}}^{\text{f.d.}} & \longrightarrow & \dots & \longrightarrow & \mathcal{A}(\mathfrak{gl}_2) - \underline{\text{Mod}}^{\text{f.d.}} \\
 \downarrow & & \downarrow & & & & \downarrow \\
 U(\mathfrak{gl}_n) - \underline{\text{Mod}}^{\text{f.d.}} & \longrightarrow & U(\mathfrak{gl}_{n-1}) - \underline{\text{Mod}}^{\text{f.d.}} & \longrightarrow & \dots & \longrightarrow & U(\mathfrak{gl}_2) - \underline{\text{Mod}}^{\text{f.d.}}
 \end{array}$$

Current & Future Work

- I proved a general result similar to the construction seen in the simple example for turning Galois rings into Galois orders.
- I have a result on the regarding (generic) Gelfand-Zeitlin modules for $\mathcal{A}(\mathfrak{gl}_n)$.
- I proved that $\mathcal{A}(\mathfrak{gl}_n)$ satisfies the Gelfand-Kirillov Conjecture in settings where the alternating group satisfies Noether's Problem.
- I am working with my advisor Jonas Hartwig to give a realization of $U(\mathfrak{so}_n)$ as a Galois order.
- I am also working to describe the so-called "standard Galois order" in the setting where G is a complex reflection group.

References

- [DF02] Drozd Y.A., Futorny V.M., Ovsienko S.A. (1994) Harish-Chandra Subalgebras and Gelfand-Zetlin Modules. In: Dlab V., Scott L.L. (eds) Finite Dimensional Algebras and Related Topics. NATO ASI Series (Series C: Mathematical and Physical Sciences), vol 424. Springer, Dordrecht, ISBN 978-94-017-1556-0, https://doi.org/10.1007/978-94-017-1556-0_5
- [FO10] Futorny, V., Ovsienko, S., Galois orders in skew monoid rings, In Journal of Algebra, Volume 324, Issue 4, 2010, Pages 598-630, ISSN 0021-8693, <https://doi.org/10.1016/j.jalgebra.2010.05.006>.
(<http://www.sciencedirect.com/science/article/pii/S0021869310002267>)
- [FO14] Vyacheslav Futorny, Serge Ovsienko, Fibers of characters in Gelfand-Tsetlin categories, Trans. Amer. Math. Soc., Volume 366, Pages 4173-4208, ISSN 1088-6850, <https://doi.org/10.1090/S0002-9947-2014-05938-2>
- [Har17] Hartwig, J. T., Principal Galois Orders and Gelfand-Zeitlin modules, 2017, arXiv:1710.04186
- [Jau19] Jauch, E. C., An Alternating Analogue of $U(\mathfrak{gl}_n)$ and Its Representations, 2019, arXiv:1907.13254

Thank you. Questions?