Galois Orders

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Overview

History and Motivation

- Studying "subalgebra-algebra" pairs finds its roots in representation theory. We will be looking at "semicommutative" pairs $\Gamma \subset \mathscr{U}$ where $\mathscr{U}$ is an associative (noncommutative) $\mathbb{C}$-algebra and Γ an integral domain.
- Motivation for such pairs comes from framework of *Harish-Chandra modules* where $\mathscr{U}$ is the universal enveloping algebra of a reductive Lie algebra and Γ is the universal enveloping algebra of a Cartan subalgebra (generalized weight modules) [DFO2].
- The objects described here were originally defined and studied by Futorny and Ovsienko in [FO10] and [FO14].
- *Galois rings* and *Galois orders* form a collection of algebras that contains many important examples:
  - *Generalized Weyl algebras* (Bavula, Rosenberg '9*)
  - Universal enveloping algebra of $\mathfrak{gl}_n$
  - Shifted Yangians and Finite W-algebras

Basic Definitions

Definition 1

For a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, the *Universal enveloping algebra of* $\mathfrak{g}$ denoted $U(\mathfrak{g})$, is the following quotient of the tensor algebra of $\mathfrak{g}$:

$$U(\mathfrak{g}) = \frac{T(\mathfrak{g})}{(x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g})}$$

Definition 2

A subalgebra $\Gamma \subseteq \mathscr{U}$ is *maximal commutative* if it is not contained in any other commutative subalgebra of $\mathscr{U}$.

Galois Rings and Galois Orders

- We will follow the setting of Hartwig's from [Har17].
- Let Λ be a Noetherian closed domain, G a subgroup of $\mathrm{Aut}(\Lambda)$, and $\mathscr{M}$ a separating submonoid of $\mathrm{Aut}(\Lambda)$ with respect to G such that G acts by conjugation on it.
- Let $\Gamma := \Lambda^G$

Definition 3

Given a commutative ring R and a submonoid $\mathscr{M} \subseteq \mathrm{Aut}(R)$, we define the *smash product* as follows:

$$R\#\mathscr{M} := \{\sum_{\mu\in\mathscr{M}} a_\mu \mu \mid a_\mu \in R \text{ and finitely many } a_\mu \neq 0\},$$

with component-wise addition, and multiplication defined by $a_1\mu_1 \cdot a_2\mu_2 = (a_1\mu_1(a_2))\mu_1\mu_2$ and expanding linearly.

- Since G acts on Λ, its action naturally extends to an action on $\text{Frac}(\Lambda)$.
- As such, G acts on $\text{Frac}(\Lambda)\#\mathscr{M}$.
- We have the following diagram:

$$\begin{array}{ccccc} \Lambda & \hookrightarrow & \text{Frac}(\Lambda) & \hookrightarrow & \text{Frac}(\Lambda)\#\mathscr{M} \\ \uparrow & & \uparrow & & \uparrow \\ \Gamma & \hookrightarrow & \text{Frac}(\Gamma) & \hookrightarrow & (\text{Frac}(\Lambda)\#\mathscr{M})^G \end{array}$$

- Note: $\text{Frac}(\Lambda)/\text{Frac}(\Gamma)$ is a Galois extension with Galois group G.

Definition 4

For an element $X \in (\text{Frac}(\Lambda)\#\mathscr{M})^G$ of the form $X = \sum a_\mu \mu$, we define $\text{supp}_{\mathscr{M}}(X) = \{\mu \mid a_\mu \neq 0\}$.

Definition 5 (Futorny-Ovsienko 2010)

A *Galois* Γ*-ring* is a subalgebra $\mathscr{U}$ of $(\text{Frac}(\Lambda)\#\mathscr{M})^G$ containing Γ such that $\text{Frac}(\Gamma)\mathscr{U} = \mathscr{U}\,\text{Frac}(\Gamma) = (\text{Frac}(\Lambda)\#\mathscr{M})^G$.

We have the following criterion for Galois rings:

Proposition 6 (Furtony-Ovsienko 2010)

Let $\mathscr{X} \subseteq (\text{Frac}(\Lambda)\#\mathscr{M})^G$ *and let* $\mathscr{U}$ *the the subring of* $(\text{Frac}(\Lambda)\#\mathscr{M})^G$ *generated by* $\Gamma \cup \mathscr{X}$*. Then* $\mathscr{U}$ *is a Galois* Γ*-ring iff* $\cup_{X\in\mathscr{X}} \text{supp}_{\mathscr{M}}(X)$ *generates* $\mathscr{M}$ *as a monoid.*

Definition 7 (Futorny-Ovsienko 2010)

A *Galois* Γ*-order* is a Galois Γ-ring such that for any finite dimensional left (or right) $\mathrm{Frac}(\Gamma)$-subspace W of $(\mathrm{Frac}(\Lambda)\#\mathscr{M})^G$, $\mathscr{U} \cap W$ is a finite generated left (resp. right) Γ-module.

- The above condition is very technical and difficult to show.
- In 2017, Hartwig showed the following condition implies the above condition:

Definition 8 (Hartwig 2017)

Let $\mathscr{U}$ be a Galois Γ-ring such that $X(\Gamma) \subseteq \Gamma$ for every $X \in \mathscr{U}$. Then $\mathscr{U}$ is a principal Galois Γ-order

- Note: Γ is maximal commutative in any Galois Γ-order.

Definition 9

A $\mathscr{U}$-module V is a *Gelfand-Zeitlin* module (with respect to Γ) if $\dim(\Gamma.v) < \infty$ for all $v \in V$.

The major results in [FO14] give:

1. The existence of "generic" simple Gelfand-Zeitlin modules over Galois rings.
2. A "rough" classification of simple Gelfand-Zeitlin modules over Galois orders.

A Simple Example

- Let $\Lambda = \mathbb{C}[x]$, $\delta \in \mathrm{Aut}(\Lambda)$ such that $\delta(x) = x - 1$, $\mathscr{M} = \langle \delta \rangle_{\mathrm{grp}} \cong \mathbb{Z}$, and G the trivial group.
- $f(x) \in \mathbb{C}[x]$ such that $f(0) \neq 0$
- Define $X, Y \in \mathrm{Frac}(\Lambda)\#\mathscr{M}$ such that

$$X := \delta \frac{f(x)}{x} \qquad \text{and} \qquad Y := \delta^{-1}.$$

- Let $\mathscr{U}_f = \mathbb{C}\langle \Lambda, X, Y \rangle_{\mathrm{alg}}$.
- Then $\mathscr{U}_f$ is a Galois Λ-ring by the Galois ring criterion because $\mathrm{supp}_{\mathscr{M}} X \cup \mathrm{supp}_{\mathscr{M}} Y = \{\delta, \delta^{-1}\}$ which generate $\mathscr{M}$.

Is $\mathscr{U}_f$ a Galois Λ-order?

- Let $C := C_{\mathscr{U}_f}(\Lambda) = \{u \in \mathscr{U}_f \mid fu = uf \ \forall f \in \Lambda\}$, the centralizer of Λ in $\mathscr{U}_f$.
- Since $YX = \dfrac{f(x)}{x} \in C \setminus \Lambda$, Λ is not maximal commutative in $\mathscr{U}_f$.
- Since Λ is not maximal commutative, $\mathscr{U}_f$ is not a Galois Λ-order.
- However, C is maximal commutative.

Question 2.1

Is $\mathscr{U}_f$ a Galois C-order?

- We first need to describe C.

Describing C

- The following lemmas help us to describe C.

Lemma 10

For any $f(x)$ such that $f(0) \neq 0$, $\frac{1}{x}, \frac{1}{x-1} \in C$.

Lemma 11

For any $f(x)$ such that $f(0) \neq 0$ and $k \geq 1$, $\frac{1}{x+k} \in C$.

Lemma 12

For any $f(x)$ such that $f(0) \neq 0$ and $k \geq 2$, $\frac{1}{x-k} \in C$.

Proposition 13 (J∗ 2019)

If $f(x)$ is a polynomial such that $f(0) \neq 0$, then

$$C = \mathbb{C}[x]\left[\frac{1}{x+k} \;\middle|\; k \in \mathbb{Z}\right].$$

Theorem 14 (J∗ 2019)

If $f(x)$ is a polynomial such that $f(0) \neq 0$, then $\mathscr{U}_f$ is a (co-)principal Galois C-order.

Set-Up

- We recall that $U(\mathfrak{gl}_n)$-modules can be represented by Gelfand-Tsetlin patterns.

Example 15

Let $(\lambda_{21}, \lambda_{22})$ be a $U(\mathfrak{gl}_2)$ weight. Then the following is a Gelfand-Tsetlin pattern $L(\lambda_{21}, \lambda_{22})$:

$$\begin{array}{cc} \boxed{\lambda_{21}} & \boxed{\lambda_{22}} \\ \multicolumn{2}{c}{\boxed{\lambda_{11}}} \end{array}$$

where $\lambda_{21} \geq \lambda_{11} \geq \lambda_{22}$.

- $U(\mathfrak{gl}_2)$ acts on these patterns via rational functions in the entries λ_{ij} and integral shifts.

- In 2010, Futorny and Ovsienko gave a realization of $U(\mathfrak{gl}_n)$ as a Galois order with $\Lambda = \mathbb{C}[x_{ki} \mid 1 \leq i \leq k \leq n]$, $G = S_1 \times S_2 \times \cdots \times S_n$, and $\mathscr{M} = \langle \delta^{j\ell} \mid 1 \leq \ell \leq j \leq n-1 \rangle_{\text{grp}}$ where $\delta^{j\ell}(x_{ki}) = x_{ki} - \delta_{jk}\delta_{i\ell}$.
- We will use U_n to denote the image of $U(\mathfrak{gl}_n)$ under the embedding $\varphi\colon U(\mathfrak{gl}_n) \hookrightarrow (\mathrm{Frac}(\Lambda)\#\mathscr{M})^G$ defined as follows:

$$\varphi(E_k^{\pm}) = \sum_{i=1}^{k} (\delta^{ki})^{\pm 1} a_{ki}^{\pm} \quad \text{with} \quad a_{ki}^{\pm} = \mp \frac{\prod_{j=i}^{k\pm 1} x_{k\pm 1,j} - x_{ki}}{\prod_{j\neq i} x_{kj} - x_{ki}},$$

$$\varphi(E_{kk}) = \sum_{i=1}^{k} (x_{ki} + i - 1) - \sum_{i=1}^{k-1} (x_{(k-1)i} + i - 1).$$

Example 16

For $n = 2$, U_2 is contained in

$$(\mathbb{C}(x_{11}, x_{21}, x_{22}) \# \langle \delta^{11} \rangle_{\mathrm{grp}})^{S_1 \times S_2}$$

where δ^{11} is an automorphism of $\mathbb{C}(x_{11}, x_{21}, x_{22})$ defined by

$$\delta^{11}(x_{ij}) = x_{ij} - \delta_{1i}\delta_{1j}.$$

with generators:

$$\begin{aligned}
\varphi(E_1^+) &= -\delta^{11}(x_{21} - x_{11})(x_{22} - x_{11}),\\
\varphi(E_1^-) &= (\delta^{11})^{-1},\\
\varphi(E_{11}) &= x_{11},\\
\varphi(E_{22}) &= x_{21} + x_{22} - x_{11} + 1.
\end{aligned}$$

- What happens when we change the symmetric groups to alternating groups?
- Recall that $\mathbb{C}[x_1, x_2, \ldots, x_n]^{A_n} = \mathbb{C}[x_1, x_2, \ldots, x_n]^{S_n}[\mathcal{V}]$.

Definition 17 (J∗ 2019)

The *alternating analogue of* $U(\mathfrak{gl}_n)$, denoted $\mathscr{A}(\mathfrak{gl}_n)$, is defined as the subalgebra of $(\mathrm{Frac}(\Lambda)\#\mathscr{M})^{A_1 \times A_2 \times \cdots \times A_n}$ generated by $U_n \cup \{\mathcal{V}_2, \mathcal{V}_3, \cdots, \mathcal{V}_n\}$ where:

$$\mathcal{V}_k = \mathcal{V}_k(x_{k1}, \ldots, x_{kk}) = \prod_{i<j}(x_{ki} - x_{kj}) \quad \text{for } k = 1, \ldots, n-1.$$

- The following proposition lists some basic properties of $\mathscr{A}(\mathfrak{gl}_n)$.

Proposition 18 (J∗ 2019)

(i) $U(\mathfrak{gl}_n) \cong U_n \subset \mathscr{A}(\mathfrak{gl}_n)$,

(ii) $\mathscr{A}(\mathfrak{gl}_n)$ *is a Galois ring,*

(iii) $\mathcal{V}_n$ *is central in* $\mathscr{A}(\mathfrak{gl}_n)$,

(iv) $Z(\mathscr{A}(\mathfrak{gl}_n)) \cong \mathbb{C}[x_1, \ldots, x_n]^{A_n}$,

(v) *there is a chain of subalgebras* $\mathscr{A}(\mathfrak{gl}_1) \subset \mathscr{A}(\mathfrak{gl}_2) \subset \cdots \subset \mathscr{A}(\mathfrak{gl}_n)$,

(vi) $\mathscr{A}(\mathfrak{gl}_n)$ *is the minimal extension of* $U(\mathfrak{gl}_n)$ *with properties* (iv) *and* (v).

The case $n = 2$

Proposition 19 (J∗ 2019)

$\mathscr{A}(\mathfrak{gl}_2)$ is isomorphic to the following extension of $U(\mathfrak{gl}_2)$:

$$\frac{U(\mathfrak{gl}_2)[T]}{\big(T^2 - (-c_{21}^2 + 2c_{22} + 1)\big)}$$

where c_{2i} are the so-called Gelfand invariants for $\mathfrak{gl}_2$ with

$$c_{21} = E_{11} + E_{22} \qquad \textit{and} \qquad c_{22} = E_{11}^2 + E_{22}^2 + E_{21}E_{12} + E_{12}E_{21}.$$

Theorem 20 (J∗, 2019)

$\mathscr{A}(\mathfrak{gl}_2)$ *is a Galois order.*

Proof idea.

We use the previous proposition, $\mathcal{V}_2$ is central, and a theorem of Futorny and Ovsienko from [FO10].

Larger n

- Unfortunately the following example shows that $\mathscr{A}(\mathfrak{gl}_n)$ is not a Galois order for $n \geq 3$.

Example 21

Let $[\cdot,\cdot]$ denote the commutator bracket. Then

$$\Big(\varphi(E_2^+) + [\varphi(E_2^+), \mathcal{V}_2]\big) \cdot \big(\varphi(E_2^-) - [\varphi(E_2^-), \mathcal{V}_2]\Big) = \delta^{21} a_{21}^+ \cdot (\delta^{21})^{-1} a_{21}^-,$$

an element centralizer of Γ in $\mathscr{A}(\mathfrak{gl}_n)$.

For $n = 2$

- The structure of $\mathscr{A}(\mathfrak{gl}_n)$-modules is related to $U(\mathfrak{gl}_n)$-modules in an interesting way.

Proposition 22 (J∗ 2019)

The finite-dimensional simple $\mathscr{A}(\mathfrak{gl}_2)$-modules are characterized by ordered pairs $(\lambda_2, \varepsilon_2)$, where $\lambda_2 := (\lambda_{21}, \lambda_{22}) \in \mathbb{C}^2$ is a dominant integral weight for $U(\mathfrak{gl}_2)$ and $\varepsilon_2 \in \{1, -1\}$.

- We note that $\mathcal{V}_2^2$ must act diagonally on any finite-dimensional $\mathscr{A}(\mathfrak{gl}_2)$-module V.
- $\mathrm{Res}_{U(\mathfrak{gl}_2)}^{\mathscr{A}(\mathfrak{gl}_2)} V$. is a direct sum of simple $U(\mathfrak{gl}_2)$-modules and $\mathcal{V}_2^2$ is a quadratic polynomial of Gelfand invariants in $U(\mathfrak{gl}_2)$.

Example 23

Let $V = V(0) \oplus V(0)$ where $U(\mathfrak{gl}_2)$ acts trivially. This means that $\mathcal{V}_2^2$ must act as Id_V. We define the following action of $\mathcal{V}_2$

$$\mathcal{V}_2 . \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

with $0 \neq \alpha \in \mathbb{C}$. It is clear then that $\mathcal{V}_2^2$ acts as the identity on V, but the subrepesentation $W = \{(v_1, 0) \mid v_1 \in V(0)\}$ is not a direct summand of V as a $\mathscr{A}(\mathfrak{gl}_2)$-module.

Larger n

Theorem 24 (J∗ 2019)

Every finite-dimensional simple module over $\mathscr{A}(\mathfrak{gl}_n)$, on which $\mathcal{V}_2, \ldots, \mathcal{V}_{n-1}$ act diagonally, is of the form $V(\lambda_n, \varepsilon_n, \varepsilon_{n-1}, \ldots, \varepsilon_2)$ where $\lambda_n = (\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nn})$ is a weight of $U(\mathfrak{gl}_n)$, $\varepsilon_j \in \{\pm 1\}^{r_{\lambda_n, j}}$, with $r_{\lambda_n, j}$ denoting the number of ways to fill the j-th row of Gelfand-Tsetlin pattern with fixed top row λ_n, and $j = 2, 3, \ldots, n$.

Proof idea.

Follows by induction on n, and the following commutative diagram:

$$\begin{array}{ccccccc}
\mathscr{A}(\mathfrak{gl}_n)\text{-}\underline{\mathrm{Mod}}^{\text{f.d.}} & \longrightarrow & \mathscr{A}(\mathfrak{gl}_{n-1})\text{-}\underline{\mathrm{Mod}}^{\text{f.d.}} & \longrightarrow & \cdots & \longrightarrow & \mathscr{A}(\mathfrak{gl}_2)\text{-}\underline{\mathrm{Mod}}^{\text{f.d.}} \\
\downarrow & & \downarrow & & & & \downarrow \\
U(\mathfrak{gl}_n)\text{-}\underline{\mathrm{Mod}}^{\text{f.d.}} & \longrightarrow & U(\mathfrak{gl}_{n-1})\text{-}\underline{\mathrm{Mod}}^{\text{f.d.}} & \longrightarrow & \cdots & \longrightarrow & U(\mathfrak{gl}_2)\text{-}\underline{\mathrm{Mod}}^{\text{f.d.}}
\end{array}$$

Current & Future Work

- I proved a general result similar to the construction seen in the simple example for turning Galois rings into Galois orders.
- I have a result on the regarding (generic) Gelfand-Zeitlin modules for $\mathscr{A}(\mathfrak{gl}_n)$.
- I proved that $\mathscr{A}(\mathfrak{gl}_n)$ satisfies the Gelfand-Kirillov Conjecture in settings where the alternating group satisfies Noether's Problem.
- I am working with my advisor Jonas Hartwig to give a realization of $U(\mathfrak{so}_n)$ as a Galois order.
- I am also working to describe the so-called "standard Galois order" in the setting where G is a complex reflection group.

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Thank you. Questions?