An Alternating Analogue of $U(\mathfrak{gl}_n)$ and Its Representations

Erich C. Jauch

Ph.D Candidate

Iowa State University

ecjauch@iastate.edu

AMS Fall Central Sectional Meeting

September 2019

Overview

1. Background
   - Some Definitions
   - $U(\mathfrak{gl}_n)$ as a Galois order
2. An Alternating Analogue of $U(\mathfrak{gl}_n)$
   - The Definition
   - Some Properties
3. Representations
   - Finite-Dimensional Simple Modules

Definitions

- We will follow the setting of Hartwig's from [Har17].
- Let Λ be a Noetherian closed domain, G a subgroup of $\mathrm{Aut}(\Lambda)$, and $\mathscr{M}$ a separating submonoid of $\mathrm{Aut}(\Lambda)$ with respect to G such that G acts by conjugation on it.

Definition 1 (Futorny-Ovsienko, 2010)

A *Galois Λ^G-ring* is a subalgebra U of $(\mathrm{Frac}(\Lambda)\#\mathscr{M})^G$ containing Λ^G such that $\mathrm{Frac}(\Lambda^G)U = U\,\mathrm{Frac}(\Lambda^G) = (\mathrm{Frac}(\Lambda)\#\mathscr{M})^G$.

Definition 2 (Futorny-Ovsienko, 2010)

A *Galois Λ^G-order* is a Galois Λ^G-ring such that for any finite dimensional left (or right) $\mathrm{Frac}(\Lambda^G)$-subspace W of $(\mathrm{Frac}(\Lambda)\#\mathscr{M})^G$, $U \cap W$ is a finite generated left (resp. right) Λ^G-module.

- Examples include: $U(\mathfrak{gl}_n)$, Generalized Weyl algebras, and finite W-algebras.

Motivation

- We recall that $U(\mathfrak{gl}_n)$-modules can be represented by Gelfand-Tsetlin patterns.

Example 3

Let $(\lambda_{21}, \lambda_{22})$ be a $U(\mathfrak{gl}_2)$ weight. Then the following is a Gelfand-Tsetlin pattern $L(\lambda_{21}, \lambda_{22})$:

$$\begin{array}{cc} \boxed{\lambda_{21}} & \boxed{\lambda_{22}} \\ \multicolumn{2}{c}{\boxed{\lambda_{11}}} \end{array}$$

where $\lambda_{21} \geq \lambda_{11} \geq \lambda_{22}$.

- $U(\mathfrak{gl}_2)$ acts on these pattern via rational functions in the entries λ_{ij} and integral shifts.

- In 2010, Futorny and Ovsienko gave a realization of $U(\mathfrak{gl}_n)$ as a Galois order with $\Lambda = \mathbb{C}[x_{ki} \mid 1 \leq i \leq k \leq n]$, $G = S_1 \times S_2 \times \cdots \times S_n$, and $\mathscr{M} = \langle \delta^{j\ell} \mid 1 \leq \ell \leq j \leq n-1 \rangle_{\text{grp}}$ where $\delta^{j\ell}(x_{ki}) = x_{ki} - \delta_{jk}\delta_{i\ell}$.
- We will use U_n to denote the image of $U(\mathfrak{gl}_n)$ under the embedding $\varphi\colon U(\mathfrak{gl}_n) \hookrightarrow (\text{Frac}(\Lambda)\#\mathscr{M})^G$.
- For $n = 2$, the algebra is

$$(\mathbb{C}(x_{11}, x_{21}, x_{22})\#\langle \delta^{11} \rangle_{\text{grp}})^{S_1 \times S_2}$$

where δ^{11} is an automorphism of $\mathbb{C}(x_{11}, x_{21}, x_{22})$ defined by

$$\delta^{11}(x_{ij}) = x_{ij} - \delta_{1i}\delta_{1j}.$$

- What happens when we change the symmetric groups to alternating groups?
- Recall that $\mathbb{C}[x_1, x_2, \ldots, x_n]^{A_n} = \mathbb{C}[x_1, x_2, \ldots, x_n]^{S_n}[\mathcal{V}]$.

Definition 4 (J∗, 2019)

The *alternating analogue of* $U(\mathfrak{gl}_n)$, denoted $\mathscr{A}(\mathfrak{gl}_n)$, is defined as the subalgebra of $(\mathrm{Frac}(\Lambda)\#\mathscr{M})^{A_1 \times A_2 \times \cdots \times A_n}$ generated by $U_n \cup \{\mathcal{V}_2, \mathcal{V}_3, \cdots, \mathcal{V}_n\}$ where:

$$\mathcal{V}_k = \mathcal{V}_k(x_{k1}, \ldots, x_{kk}) = \prod_{i<j}(x_{ki} - x_{kj}) \quad \text{for } k = 1, \ldots, n-1.$$

- The following proposition lists some basic properties of $\mathscr{A}(\mathfrak{gl}_n)$.

Proposition 5 (J∗, 2019)

(i) $U(\mathfrak{gl}_n) \cong U_n \subset \mathscr{A}(\mathfrak{gl}_n)$,

(ii) $\mathscr{A}(\mathfrak{gl}_n)$ *is a Galois ring,*

(iii) $\mathcal{V}_n$ *is central in* $\mathscr{A}(\mathfrak{gl}_n)$,

(iv) $Z(\mathscr{A}(\mathfrak{gl}_n)) \cong \mathbb{C}[x_1, \ldots, x_n]^{A_n}$,

(v) *there is a chain of subalgebras* $\mathscr{A}(\mathfrak{gl}_1) \subset \mathscr{A}(\mathfrak{gl}_2) \subset \cdots \subset \mathscr{A}(\mathfrak{gl}_n)$,

(vi) $\mathscr{A}(\mathfrak{gl}_n)$ *is the minimal extension of* $U(\mathfrak{gl}_n)$ *with properties iv and v.*

The case $n = 2$

Proposition 6 (J∗, 2019)

$\mathscr{A}(\mathfrak{gl}_2)$ is isomorphic to the following extension of $U(\mathfrak{gl}_2)$:

$$\frac{U(\mathfrak{gl}_2)[T]}{\left(T^2 - (-c_{21}^2 + 2c_{22} + 1)\right)}$$

where c_{2i} are the so-called Gelfand invariants for $\mathfrak{gl}_2$ with

$$c_{21} = E_{11} + E_{22} \qquad \text{and} \qquad c_{22} = E_{11}^2 + E_{22}^2 + E_{21}E_{12} + E_{12}E_{21}.$$

Theorem 7 (J∗, 2019)

$\mathscr{A}(\mathfrak{gl}_2)$ is a Galois order.

For $n = 2$

- The structure of $\mathscr{A}(\mathfrak{gl}_n)$-modules is related to $U(\mathfrak{gl}_n)$-modules in an interesting way.

Proposition 8 (J∗, 2019)

The finite-dimensional simple $\mathscr{A}(\mathfrak{gl}_2)$-modules are characterized by ordered pairs $(\lambda_2, \varepsilon_2)$, where $\lambda_2 := (\lambda_{21}, \lambda_{22}) \in \mathbb{C}^2$ is a dominant integral weight for $U(\mathfrak{gl}_2)$ and $\varepsilon_2 \in \{1, -1\}$.

Larger n

Theorem 9 (J∗, 2019)

Every finite-dimensional simple module over $\mathscr{A}(\mathfrak{gl}_n)$, on which $\mathcal{V}_2, \ldots, \mathcal{V}_{n-1}$ act diagonally, is of the form $V(\lambda_n, \varepsilon_n, \varepsilon_{n-1}, \ldots, \varepsilon_2)$ where $\lambda_n = (\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nn})$ is a weight of $U(\mathfrak{gl}_n)$, $\varepsilon_j \in \{\pm 1\}^{r_{\lambda_n,j}}$, with $r_{\lambda_n,j}$ denoting the number of ways to fill the j-th row of Gelfand-Tsetlin pattern with fixed top row λ_n, and $j = 2, 3, \ldots, n$.

Proof idea.

Follows by induction on n, and the following commutative diagram:

$$\begin{array}{ccccccc}
\mathscr{A}(\mathfrak{gl}_n)\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}} & \longrightarrow & \mathscr{A}(\mathfrak{gl}_{n-1})\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}} & \longrightarrow & \cdots & \longrightarrow & \mathscr{A}(\mathfrak{gl}_2)\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}} \\
\downarrow & & \downarrow & & & & \downarrow \\
U(\mathfrak{gl}_n)\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}} & \longrightarrow & U(\mathfrak{gl}_{n-1})\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}} & \longrightarrow & \cdots & \longrightarrow & U(\mathfrak{gl}_2)\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}}
\end{array}$$

References

[FO10] Futorny, V., Ovsienko, S., Galois orders in skew monoid rings, In Journal of Algebra, Volume 324, Issue 4, 2010, Pages 598-630, ISSN 0021-8693, https://doi.org/10.1016/j.jalgebra.2010.05.006. (http://www.sciencedirect.com/science/article/pii/S0021869310002267)

[Har17] Hartwig, J. T., Principal Galois Orders and Gelfand-Zeitlin modules, 2017, arXiv:1710.04186

[Jau19] Jauch, E. C., An Alternating Analogue of $U(\mathfrak{gl}_n)$ and Its Representations, 2019, arXiv:1907.13254

Thank you. Questions?