Galois orders & an alternating analogue of $U(\mathfrak{gl}_n)$

Erich C. Jauch

Iowa State University, USA

Introduction

Galois rings and Galois orders were intially defined and studies by V. Futorny and S. Ovsienko in [1] and [2]. They form a class of algebras that contain many important examples including: Generalized Weyl Algebras, the universal enveloping of $\mathfrak{gl}_n$, and shifted Yangians. In [1] is a description of $U(\mathfrak{gl}_n)$ as a Galois order with relation to a direct product of symmetric groups. The goal of this project is to study a similar object using a direct product of alternating groups.

Key Assumptions from [3]

Let Λ be a integrally closed domain, G a finite subgroup of $\mathrm{Aut}(\Lambda)$, $\mathscr{M}$ a submonoid of $\mathrm{Aut}(\Lambda)$

(A1) $(\mathscr{M}\mathscr{M}^{-1}) \cap G = 1_{\mathrm{Aut}(\Lambda)}$ *(separation)*

(A2) $\forall g \in G, \forall \mu \in \mathscr{M}$: ${}^{g}\mu = g \circ \mu \circ g^{-1} \in \mathscr{M}$ *(invariance)*

(A3) Λ is noetherian as a module over Λ^G *(finiteness)*

General Construction

Given such a Λ, $\mathscr{M}$, and G, we construct the following: $L = \mathrm{Frac}\,\Lambda$ (Note that $\mathscr{M}$ and G act naturally on L by automorphisms), $\mathscr{L} = \mathscr{M} * L$ the skew monoid ring[1], defined as the free right L-module on $\mathscr{M}$ with multiplication given by $\mu_1 a_1 \cdot \mu_2 a_2 = (\mu_1\mu_2)(\mu_2^{-1}(a_1)a_2)$ and extending linearly. By (A2) it is clear that G acts on $\mathscr{L}$ by $g(\mu a) = {}^{g}\mu g(a)$. Let $\Gamma = \Lambda^G$, $K = L^G$ and $\mathscr{K} = \mathscr{L}^G$ the respective G-invariant subring. This leads to the following diagram of inclusions:

$$\begin{array}{ccccc} \Lambda & \longrightarrow & L & \longrightarrow & \mathscr{L} \\ \uparrow & & \uparrow & & \uparrow \\ \Gamma & \longrightarrow & K & \longrightarrow & \mathscr{K} \end{array}$$

Figure 1:

We note that in this construction it follows that $K = \mathrm{Frac}(\Gamma)$ and L/K is a Galois extension with $\mathrm{Gal}(L/K) = G$.

Definitions

i) A finitely generated Γ-subring $U \subseteq \mathscr{K}$ is called a *Galois Γ-ring* (or *Galois ring with respect to* Γ) if $KU = UK = \mathscr{K}$.

ii) A Galois Γ-ring $\mathscr{U}$ in $\mathscr{K}$ is a *left* (respectively *right*) *Galois Γ-order in* $\mathscr{K}$ if for any finite-dimensional left (respectively right) K-subspace $W \subseteq \mathscr{K}$, $W \cap \mathscr{U}$ is a finitely generated left (respectively right) Γ-module. A Galois Γ-ring $\mathscr{U}$ in $\mathscr{K}$ is a *Galois Γ-order in* $\mathscr{K}$ if $\mathscr{U}$ is a left and right Galois Γ-order in $\mathscr{K}$.

Major Result from [2]

It gives a "rough" classification of simple modules V over a Galois Γ-order $\mathscr{U}$ for which every $\dim(\Gamma.v) < \infty$ for every $v \in V$. Such modules are called *simple Gelfand-Zeitlin modules*.

Defining the Alternating Analogue

Let $\varphi\colon U(\mathfrak{gl}_n) \hookrightarrow \mathscr{K}$ be the embedding defined in [1], and $U := \varphi(U(\mathfrak{gl}_n))$. We observe that cleary $\mathscr{K} \subset \widetilde{\mathscr{K}}$, so we define our alternating analogue as the following subalgebra of $\widetilde{\mathscr{K}}$:

$$\mathsf{A}(\mathfrak{gl}_n) = \mathbb{C}\langle U, \mathcal{V}_\ell \mid 2 \le \ell \le n\rangle$$

Following standard notation we define the following:

$$X_{k,0}^{+} := \varphi(E_{k,k+1}) = \sum_{i=1}^{k} \delta^{ki} A_{ki}^{+},$$

$$X_{k,0}^{-} := \varphi(E_{k,k-1}) = \sum_{i=1}^{k} (\delta^{ki})^{-1} A_{ki}^{-},$$

$$X_{kk} := \varphi(E_{kk}) = \sum_{i=1}^{k} (x_{ki} - i + 1) - \sum_{j=1}^{k-1} (x_{(k-1)j} - j + 1),$$

$$X_{k,1}^{\pm} := \pm[\mathcal{V}_k, X_{k,0}^{\pm}] \text{ taking } \mathcal{V}_1 = 1$$

Main Result

For the algebra $\mathsf{A}(\mathfrak{gl}_n)$ defined above the following statements hold:

i) For every $n \ge 2$, $\mathsf{A}(\mathfrak{gl}_n)$ is a Galois $\widetilde{\Gamma}$-ring,

ii) $\mathsf{A}(\mathfrak{gl}_n)$ is a Galois $\widetilde{\Gamma}$-order if and only if $n = 2$

Our Construction

- $\Lambda := \mathbb{C}[x_{ki} \mid 1 \le i \le k \le n]$
- $\mathscr{M} := \mathbb{Z}^{n(n-1)/2}$ with $\delta^{ki} \in \mathscr{M}$ acting on $x_{\ell j}$ by $\delta^{ki}(x_{\ell j}) = x_{\ell j} - \delta_{k\ell}\delta_{ij}$
- $G := \mathbb{S}_n = S_1 \times S_2 \times \cdots \times S_n$, $\widetilde{G} := \mathbb{A}_n = A_1 \times A_2 \times \cdots \times A_n$
- $\Gamma := \Lambda^G = \mathbb{C}[e_{ki} \mid 1 \le i \le k \le n]$, $\widetilde{\Gamma} := \Lambda^{\widetilde{G}} = \mathbb{C}[e_{ki}, \mathcal{V}_\ell \mid 1 \le i \le k \le n, 2 \le \ell \le n]$ where e_{ki} is the ith elementary symmetric polynomial in $x_{k1}, \cdots, x_{kk}$ and $\mathcal{V}_\ell$ is the Vandermonde polynomial.
- $L = \mathrm{Frac}(\Lambda)$, $K = \mathrm{Frac}(\Gamma)$, and $\widetilde{K} = \mathrm{Frac}(\widetilde{\Gamma})$
- $\mathscr{L} = \mathscr{M} * L$, $\mathscr{K} = \mathscr{L}^G$, and $\widetilde{\mathscr{K}} = \mathscr{L}^{\widetilde{G}}$

Complete Description for $n = 2$

We know that $\mathsf{A}(\mathfrak{gl}_2)$ is a Galois $\widetilde{\Gamma}$-order; moreover, we have the following isomorphism to a cetnral extension of $U(\mathfrak{gl}_2)$.

$$\widetilde{\varphi}\colon \frac{U(\mathfrak{gl}_2)[T]}{(T^2 - (-c_{21}^2 + 2c_{22} + 1))} \xrightarrow{\cong} \mathsf{A}(\mathfrak{gl}_2),$$

given by $\widetilde{\varphi}|_{U(\mathfrak{gl}_2)} = \varphi$ from [1] and $\widetilde{\varphi}(T) = \mathcal{V}_2$. Note that c_{2i} are the Gelfand invariants.

The reason for the ease of a complete description for the $n = 2$ case stems from the general fact that $\mathcal{V}_n$ will always be central in $\mathsf{A}(\mathfrak{gl}_n)$.

Current Work

We are currently working to give a complete presentation of $\mathsf{A}(\mathfrak{gl}_3)$. We note one relation of interest below for its similarity to the Yangian realation (15) in [4]:

$$\sum_{\pi \in S_2} [X_{i,r_{\pi(1)}}, [X_{i,r_{\pi(2)}}, X_{j,s}]] = 0$$

for $|i - j| = 1$, $r_1, r_2, s \in \{0, 1\}$.

We are also working to describe all of the finite dimensional simple modules over $\mathsf{A}(\mathfrak{gl}_n)$ especially in regards to $U(\mathfrak{gl}_n)$ modules.

Acknowledgements & References

The author would like to thank his advisor Jonas T. Hartwig for his guidance. He would also like to thank the Graduate and Professional Student Senate at Iowa State University, the ISU Mathematics Department, and the organizers for helping to fund his travel to Escola de Álgebra and giving him the opportunity to present his research.

[1] Vyacheslav Futorny and Serge Ovsienko. Galois orders in skew monoid rings. *Journal of Algebra*, 324(4):598 – 630, 2010.

[2] Vyacheslav Futorny and Serge Ovsienko. Fibers of characters in gelfand-tsetlin categories. *Transactions of the American Mathematical Society*, 366:4173–4208.

[3] J. T. Hartwig. Principal Galois orders and Gelfand-Zeitlin modules. *ArXiv e-prints*, October 2017.

[4] Vyjayanthi Chari and Andrew N. Pressley. *A Guide to Quantum Groups*. Cambridge University Press, Cambridge, London, 1995.

Contact Information

- Web: https://sites.google.com/iastate.edu/ecjauch
- Email: ecjauch@iastate.edu
- Phone: +1 815 830 3757

[1] We note that this is not the standard construction of $\mathscr{L}$; however, this is the construction used for $U(\mathfrak{gl}_n)$.