An Alternating Analogue of $U(\mathfrak{gl}_n)$ and Its Representations

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Overview

1. Some History and Motivation
2. Galois Rings and Galois Orders
   - Galois Rings
   - Galois Orders
   - Co-Principal Galois Orders
   - $U(\mathfrak{gl}_n)$ as a Galois Order
3. An Alternating Analogue of $U(\mathfrak{gl}_n)$
   - Definition
   - Preserved Properties
4. Representations
   - Gefland-Zeitlin Modules
5. Future Work

History and Motivation

- These types of objects are a generalization of the framework of *Harish-Chandra modules* where $\mathscr{U}$ is the universal enveloping algebra of a reductive Lie algebra and Γ is the universal enveloping algebra of a Cartan subalgebra (generalized weight modules) [DFO2].
- The *Galois rings* and *Galois orders* were originally defined and studied by Futorny and Ovsienko in 2010.
- They form a collection of algebras that contains many important examples:
    - *Generalized Weyl algebras* defined by independently by Bavula and Rosenberg in the early nineties.
    - Universal enveloping algebra of $\mathfrak{gl}_n$
    - Finite W-algebras

Galois Rings and Galois Orders: The Setup

Let Λ be a integrally closed domain, G a finite subgroup of $\mathrm{Aut}(\Lambda)$, $\mathscr{M}$ a submonoid of $\mathrm{Aut}(\Lambda)$.

Assumptions (Hartwig 2017)

(A1) $(\mathscr{M}\mathscr{M}^{-1}) \cap G = 1_{\mathrm{Aut}(\Lambda)}$ (*separation*)

(A2) $\forall g \in G, \forall \mu \in \mathscr{M} : {}^{g}\mu = g \circ \mu \circ g^{-1} \in \mathscr{M}$ (*invariance*)

(A3) Λ is noetherian as a module over Λ^G (*finiteness*)

Example

- Let $\Lambda = \mathbb{C}[x_{11}, x_{21}, x_{22}]$ and $\mathscr{M} = \langle \delta^{11} \rangle_{grp}$ where $\delta^{11}(x_{ij}) = x_{ij} - \delta_{1i}\delta_{1j}$
- We consider $\operatorname{Frac}\Lambda = \mathbb{C}(x_{11}, x_{21}, x_{22})$, and $\mathbb{C}(x_{11}, x_{21}, x_{22}) * \langle \delta^{11} \rangle_{grp}$ which is the *skew monoid ring* the following direct sum

$$\bigoplus_{n \in \mathbb{Z}} (\delta^{11})^n \mathbb{C}(x_{11}, x_{21}, x_{22})$$

with multiplication defined by
$(\delta^{11})^n f(x_{11}, x_{21}, x_{22}) \cdot (\delta^{11})^m g(x_{11}, x_{21}, x_{22}) =$
$(\delta^{11})^{n+m} f(x_{11} + m, x_{21}, x_{22}) g(x_{11}, x_{21}, x_{22})$

Example

- Let $G = S_1 \times S_2 = \{(1)_1\} \times \{(1)_2, (12)_2\}$ with $\sigma_j(x_{ji}) = x_{j\sigma_j(i)}$.
- We look at the G invariant subrings of each of these. We denote $\Gamma = \Lambda^G$ we get the following diagram:

$$\begin{array}{ccccc}
\mathbb{C}[x_{11}, x_{21}, x_{22}] & \hookrightarrow & \mathbb{C}(x_{11}, x_{21}, x_{22}) & \hookrightarrow & \mathbb{C}(x_{11}, x_{21}, x_{22}) * \langle \delta^{11} \rangle_{grp} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{C}[x_{11}, x_{21}, x_{22}]^G & \hookrightarrow & \mathbb{C}(x_{11}, x_{21}, x_{22})^G & \hookrightarrow & (\mathbb{C}(x_{11}, x_{21}, x_{22}) * \langle \delta^{11} \rangle_{grp})^G
\end{array}$$

Definition of Galois rings

Definition (Furtorny, Ovsienko 2010)

A finitely generated Γ-subring $\mathscr{U} \subseteq (\operatorname{Frac}\Lambda * \mathscr{M})^G$ is called a *Galois* Γ*-ring* (or *Galois ring with respect to* Γ) if
$(\operatorname{Frac}\Lambda)^G \mathscr{U} = \mathscr{U}(\operatorname{Frac}\Lambda)^G = (\operatorname{Frac}\Lambda * \mathscr{M})^G$.

We have the following criterion for Galois rings

Proposition (Furtony, Ovsienko 2010)

Let $\mathscr{X} \subseteq (\operatorname{Frac}\Lambda * \mathscr{M})^G$ *and let* $\mathscr{U}$ *the the subring of* $(\operatorname{Frac}\Lambda * \mathscr{M})^G$ *generated by* $\Gamma \cup \mathscr{X}$. *Then* $\mathscr{U}$ *is a Galois* Γ*-ring iff* $\cup_{X \in \mathscr{X}} \operatorname{supp}_{\mathscr{M}}(X)$ *generates* $\mathscr{M}$ *as a monoid.*

Definition of Galois orders

Definition (Furtorny, Ovsienko 2010)

A Galois Γ-ring $\mathscr{U}$ in $(\operatorname{Frac}\Lambda * \mathscr{M})^G$ is a *left* (respectively *right*) *Galois* Γ*-order in* $(\operatorname{Frac}\Lambda * \mathscr{M})^G$ if for any finite-dimensional left (respectively right) $\operatorname{Frac}(\Lambda)^G$-subspace $W \subseteq (\operatorname{Frac}\Lambda * \mathscr{M})^G$, $W \cap \mathscr{U}$ is a finitely generated left (respectively right) Γ-module. A Galois Γ-ring $\mathscr{U}$ in $(\operatorname{Frac}\Lambda * \mathscr{M})^G$ is a *Galois* Γ*-order in* $(\operatorname{Frac}\Lambda * \mathscr{M})^G$ if $\mathscr{U}$ is a left and right Galois Γ-order in $(\operatorname{Frac}\Lambda * \mathscr{M})^G$.

Example

The subalgebra $\mathscr{U} = (\mathbb{C}[x_{11}, x_{21}, x_{22}] * \langle \delta^{11} \rangle_{grp})^{S_1 \times S_2}$ is a Galois $\mathbb{C}[x_{11}, x_{21}, x_{22}]^{S_1 \times S_2}$-order in our earlier construction.

Definition

1. Γ is *maximal commutative* in $\mathscr{U}$ if it is not properly contained in any other commutative subalgebra of $\mathscr{U}$.
2. Γ is a *Harish-Chandra subalgebra* of $\mathscr{U}$ if for every $u \in \mathscr{U}$, $\Gamma u \Gamma$ is finitely generated as both a left and right Γ-module.

We have the following criterion for Galois orders:

Proposition (Follows from Furtorny, Ovsienko 2010 Thm. 5.2)

Assume $\mathscr{U}$ is a Galois Γ-ring,

1. *If $\mathscr{U}$ is a Galois Γ-order, then Γ is maximal commutative.*
2. *If Γ is finitely generated Harish-Chandra subalgebra of $\mathscr{U}$ and $\mathscr{M}$ is a group, then the converse holds.*

Co-Evaluations

Definition (Hartwig 2017)

For $X = \sum_{\mu \in \mathscr{M}} \mu \alpha_\mu \in \operatorname{Frac} \Lambda * \mathscr{M}$ and $a \in \operatorname{Frac} \Lambda$ we define the *co-evaluation of X at a* by

$$X^\dagger(a) = \sum_{\mu \in \mathscr{M}} \alpha_\mu \cdot (\mu^{-1}(a)) \in \operatorname{Frac} \Lambda$$

Co-Principal Galois Orders

Definition (Hartwig 2017)

We define the *co-standard Galois* Γ*-order* by

$$_{\Gamma}(\operatorname{Frac}\Lambda * \mathscr{M})^G := \{X \in (\operatorname{Frac}\Lambda * \mathscr{M})^G \mid X^{\dagger}(\gamma) \in \Gamma \ \forall \gamma \in \Gamma\}.$$

Definition (Hartwig 2017)

Let $\mathscr{U}$ be a Galois Γ-ring in $(\operatorname{Frac}\Lambda * \mathscr{M})^G$. If $\mathscr{U} \subseteq {}_{\Gamma}(\operatorname{Frac}\Lambda * \mathscr{M})^G$, then is a Galois Γ-order and is called a *co-principal Galois* Γ*-order*.

$U(\mathfrak{gl}_n)$ Construction

- $\Lambda = \mathbb{C}[x_{ki} \mid 1 \leq i \leq k \leq n]$
- $\mathscr{M} = \langle \delta^{ki} \mid 1 \leq i \leq k < n \rangle_{grp}$ with the action $\delta^{ki}(x_{\ell j}) = x_{\ell j} - \delta_{k\ell}\delta_{ij}$
- $G = \mathbb{S}_n := S_1 \times S_2 \times \cdots \times S_n$
- So in this setting, $\Gamma = \mathbb{C}[e_{ki} \mid 1 \leq i \leq k \leq n]$ where $e_{ki} = \sum_{1 \leq j_1 < \cdots < j_i \leq k} x_{j_1} x_{j_2} \cdots x_{j_i}$
- In [FO10], they define an injective algebra homomorphism $\varphi \colon U(\mathfrak{gl}_n) \to (\operatorname{Frac} \Lambda * \mathscr{M})^{\mathbb{S}_n}$. We note the image of some of the generators $E_k^+ := E_{k,k+1}$ and $E_k^- := E_{k+1,k}$:

$$\varphi(E_k^{\pm}) = \mp \sum_{i=1}^{k} (\delta^{ki})^{\pm 1} \frac{\prod_{j=i}^{k\pm 1} x_{k\pm 1,j} - x_{ki}}{\prod_{j \neq i} x_{kj} - x_{ki}}$$

- $U_n := \varphi(U(\mathfrak{gl}_n))$ is a co-principal Galois Γ-order.

Our Construction

- Λ, $\mathscr{M}$ as for $U(\mathfrak{gl}_n)$.
- $\widetilde{G} = \mathbb{A}_n := A_1 \times A_2 \times \cdots \times A_n$
- $\Lambda^{\mathbb{A}_n} = \mathbb{C}[e_{ki}, \mathcal{V}_\ell \mid 1 \leq i \leq k \leq n, 2 \leq \ell \leq n]$ where $e_{ki} = \sum_{1 \leq j_1 < \cdots < j_i \leq k} x_{j_1} x_{j_2} \cdots x_{j_i}$ and $\mathcal{V}_\ell = \prod_{1 \leq i < j \leq \ell} (x_{\ell i} - x_{\ell j})$
- Let $\ell = (\ell_1, \ell_2, \ldots, \ell_{k-1}) \in \mathbb{Z}^{k-1}$. We denote a shifted Vandermonde as follows:

$$\mathcal{V}_{k,\ell} := \prod_{i<j} (x_{ki} - x_{kj} + \sum_{n=i}^{j-1} \ell_n).$$

Our Construction

- Let $S := \langle \mathcal{V}_{k,\ell} \mid \ell \in \mathbb{Z}^{k-1};\ k = 2, \ldots, n-1 \rangle^{\mathbb{A}_n}_{Monoid}$. We observe that S is a multiplicatively closed set in Λ and $\Lambda^{\mathbb{A}_n}$.
- Let $\widetilde{\Lambda} = S^{-1}\Lambda$ and $\widetilde{\Gamma} = \widetilde{\Lambda}^{\mathbb{A}_n}$
- Note that $\operatorname{Frac}\widetilde{\Lambda} = \operatorname{Frac}\Lambda$.

Defining the Alternating Analogue

Definition (J∗ 2018)

We define the *Alternating Analogue of* $U(\mathfrak{gl}_n)$ as the following subalgebra of $(\operatorname{Frac}\Lambda * \mathscr{M})^{\mathbb{A}_n}$:

$$\mathrm{A}(\mathfrak{gl}_n) := \mathbb{C}\langle U_n, \widetilde{\Gamma}\rangle_{alg}.$$

Theorem (J∗ 2018)

$\mathrm{A}(\mathfrak{gl}_n)$ *defined above is a co-principal Galois* $\widetilde{\Gamma}$*-order in* $(\operatorname{Frac}\Lambda * \mathscr{M})^{\mathbb{A}_n}$ *for all* n.

Some Properties that are Preserved

- For enveloping algebra we have the following chain of containment's $U(\mathfrak{gl}_1) \subset U(\mathfrak{gl}_2) \subset U(\mathfrak{gl}_3) \subset \cdots$. Similarly, $\mathrm{A}(\mathfrak{gl}_1) \subset \mathrm{A}(\mathfrak{gl}_2) \subset \mathrm{A}(\mathfrak{gl}_3) \cdots$
- For $U(\mathfrak{gl}_k)$ we recall that $Z(U(\mathfrak{gl}_k)) \cong \mathbb{C}[x_1, \cdots, x_k]^{S_k}$. We have a similar phenomenon $Z(\mathrm{A}(\mathfrak{gl}_k)) \cong \mathbb{C}[x_1, \cdots, x_k]^{A_k}$
- We have a complete presentation in relation to $U(\mathfrak{gl}_2)$.

Presentation for $n = 2$

Theorem (J∗ 2018)

$\mathrm{A}(\mathfrak{gl}_2)$ *is isomorphic to the following extension of* $U(\mathfrak{gl}_2)$*:*

$$\frac{U(\mathfrak{gl}_2)[T_2]}{\left(T_2^2 - (-c_{21}^2 + 2c_{22} + 1)\right)}$$

where c_{2i} *are the so-called Gelfand invariants for* $\mathfrak{gl}_2$ *with*

$$c_{21} = E_{11} + E_{22} \qquad \text{and} \qquad c_{22} = E_{11}^2 + E_{22}^2 + E_{21}E_{12} + E_{12}E_{21}.$$

Defined by φ^{-1} *when restricted to* U_2 *and mapping* $\mathcal{V}_2$ *to* T_2*.*

Some Known Results for Gelfand-Zeitlin Modules

- We assume Λ is finitely generated over $\Bbbk$ with $\overline{\Bbbk} = \Bbbk$ and char $\Bbbk = 0$

Definition

For a Galois Γ-ring $\mathscr{U}$. A left $\mathscr{U}$-module V is a *Gelfand-Zeitlin* module (with respect to Γ) if $\dim(\Gamma.v) < \infty$ for all $v \in V$. Equivalently,

$$V = \bigoplus_{\xi \in \hat{\Gamma}} V_\xi, \qquad V_\xi = \{v \in V \mid (\ker \xi)^N v = 0, \ N \gg 0\}.$$

Where $\hat{\Gamma}$ is the set of all Γ-characters.

- For $\xi \in \hat{\Gamma}$, the *fiber* $\phi(\xi)$ is set of isoclasses of simple GZ-modules V with $V_\xi \neq 0$.

Definition

Theorem (Futorny, Ovsienko 2014)

Let Γ be a commutative domain which is finitely generated as a $\Bbbk$-algebra, $\mathscr{U}$ a Galois Γ-Order, and $\xi \in \hat{\Gamma}$. Suppose that $\mathscr{M}$ is a group.

- *The fiber $\phi(\xi)$ is nonempty and finite.*

Theorem (Hartwig 2017)

Let $\xi \in \hat{\Gamma}$ be any character. If $\mathscr{U}$ is a co-principal Galois Γ-order in $(\mathrm{Frac}(\Lambda) * \mathscr{M})^G$, then the left cyclic $\mathscr{U}$-module $\mathscr{U}\xi$ has a unique simple quotient $V'(\xi)$. Moreover, $V'(\xi)$ is a Gelfand-Zeitlin module over $\mathscr{U}$ with $V'(\xi)_\xi \neq 0$ and is called a canonical Gelfand-Zeitlin module.

- This issue for us is that $\widetilde{\Lambda}$ and $\widetilde{\Gamma}$ are not finitely generated $\mathbb{C}$-algebras for $n \geq 3$.

Canonical GZ modules for $A(\mathfrak{gl}_n)$

- We can still describe canonical GZ modules for some characters ξ.
- Namely, ξ such that $\ker \xi = S^{-1}\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of $\Lambda^{\mathbb{A}_n}$ such that $\mathfrak{m} \cap S = \emptyset$. That is $\ker \xi$ is an extension of a maximal ideal of $\Lambda^{\mathbb{A}_n}$.
- For such ξ, the arguments of the previous theorem still apply. So we get unique simple quotients $V'(\xi)$ for these ξ.
- For $n = 2$ even more can be said.

$\mathrm{A}(\mathfrak{gl}_2)$ and its Representations

- Because here $\widetilde{\Lambda} = \Lambda$ is a finitely generated $\mathbb{C}$-algebra, we can apply all of the previous results.
- It is also relatively straight forward to characterize the finite dimensional simple $\mathrm{A}(\mathfrak{gl}_2)$ modules.

Theorem (J∗ 2018)

The finite dimensional simple $\mathrm{A}(\mathfrak{gl}_2)$ modules are characterized by triples $(\lambda_1, \lambda_2, \varepsilon)$ where $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ is a weight of a $U(\mathfrak{gl}_2)$ module and $\varepsilon \in \{1, -1\}$. Where ϵ denotes the sign of $\mathcal{V}_2$ acting on the standard tableau basis of a finite dimensional irreducible $U(\mathfrak{gl}_2)$ module.

- This follows from the realization that $\mathrm{A}(\mathfrak{gl}_2) \cong U(\mathfrak{gl}_2) \oplus T_2 U(\mathfrak{gl}_2)$ as a right $U(\mathfrak{gl}_2)$-module.

Future Work

- Find a presentation for $\mathrm{A}(\mathfrak{gl}_n)$ for $n \geq 3$ (Connection to $Y_2(\mathfrak{gl}_n)$).
- Describe the finite dimensional simple modules of $\mathrm{A}(\mathfrak{gl}_n)$ for $n \geq 3$.
  - Induction Argument
  - Induced and Restricted modules with $U(\mathfrak{gl}_n)$
- Characterize the relation of $U(\mathfrak{gl}_n)$-mod and $\mathrm{A}(\mathfrak{gl}_n)$-mod.
- Characterize the Canonical GZ modules for ξ where $\ker \xi$ is not an extended maximal ideal (or show that no such ξ exist).

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Thank you. Questions?