

Maps between standard and principal flag orders

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History and Motivation

- These types of objects are a generalization of the framework of *Harish-Chandra modules* where \mathcal{U} is the universal enveloping algebra of a reductive Lie algebra and Γ is the universal enveloping algebra of a Cartan subalgebra (generalized weight modules) [DFO94].
- *Galois orders* were originally defined and studied by Futorny and Ovsienko in [FO10].
- They form a collection of algebras that contains many important examples:
 - *Generalized Weyl algebras* defined by independently by Bavula and Rosenberg in the early nineties.
 - Universal enveloping algebra of \mathfrak{gl}_n
 - Finite W -algebras
- This unifies the study of their Gelfand-Tsetlin modules [FO14].

Galois Orders

- Let Λ be a Noetherian integrally closed domain, G a finite subgroup of $\text{Aut } \Lambda$, and \mathcal{M} a submonoid of $\text{Aut } \Lambda$, normalized by G , such that $\mathcal{M}\mathcal{M}^{-1} \cap G = 1$. The triple $(\Lambda, G, \mathcal{M})$ is called a *Galois order datum*.
- Let $\Gamma := \Lambda^G$

Definition 1

Given a commutative ring R and a submonoid $\mathcal{M} \subseteq \text{Aut } R$, we define the *smash product* as follows:

$$R \# \mathcal{M} := \left\{ \sum_{\mu \in \mathcal{M}} a_{\mu} \mu \mid a_{\mu} \in R \text{ and finitely many } a_{\mu} \neq 0 \right\},$$

with component-wise addition, and multiplication defined by $a_1 \mu_1 \cdot a_2 \mu_2 = (a_1 \mu_1(a_2)) \mu_1 \mu_2$ and extending bilinearly.

- Since G acts on Λ , its action naturally extends to an action on $\text{Frac}(\Lambda)$.
- As such, G acts on $\text{Frac}(\Lambda)\#\mathcal{M}$:
- Galois Γ -orders are particular subalgebras of $(\text{Frac}(\Lambda)\#\mathcal{M})^G$ containing Γ .
- In [Har20], Hartwig showed the following

Theorem 2 (Hartwig)

Let \mathcal{U} be a Galois Γ -ring such that $X(\Gamma) \subseteq \Gamma$ for every $X \in \mathcal{U}$. Then \mathcal{U} is a Galois Γ -order.

- A Galois Γ -ring satisfying this condition is called a *principal Galois Γ -order*.
- Note: Γ is maximal commutative in any Galois Γ -order.

Example 3

$(\Lambda\#\mathcal{M})^G$ is a (principal) Galois Γ -order.

Flag Orders

- In [Web19], Webster introduced principal flag orders.
- The *flag order datum* is a triple $(\Lambda, W, \mathcal{M})$ under the same restrictions as before (Replace G with W from the Galois order datum).
- Instead of looking at invariants, however, we look at $\text{Frac}(\Lambda) \# (W \times \mathcal{M})$.
- Sometimes $W \times \mathcal{M}$ is denoted by \hat{W} for shorter notation.

Definition 4

A *principal flag order* with data $(\Lambda, W, \mathcal{M})$ is a subalgebra $F \subset \text{Frac}(\Lambda) \# (W \ltimes \mathcal{M})$ such that:

- (i) $\Lambda \# W \subset F$,
- (ii) $\text{Frac}(\Lambda)F = \text{Frac}(\Lambda) \# (W \ltimes \mathcal{M})$,
- (iii) For every $X \in F$, $X(\Lambda) \subset \Lambda$.

Definition 5

The *standard flag order* with data $(\Lambda, W, \mathcal{M})$ is the subalgebra of all elements X in $\text{Frac}(\Lambda) \# (W \ltimes \mathcal{M})$ satisfying (iii) and is denoted \mathcal{F}_Λ .

Example 6

If $\Lambda = \mathbb{C}[x_1, x_2, \dots, x_n]$ and \hat{W} is a finite complex reflection group acting on \mathbb{C}^n , then \mathcal{F}_Λ is the nilHecke algebra of \hat{W} (see [Web19]).

- What follows are my results from [Jau22]
- Let $(\Lambda_1, W_1, \mathcal{M}_1)$, $(\Lambda_2, W_2, \mathcal{M}_2)$ be two flag order data, and $\text{Frac}(\Lambda_i)$ the field of fractions of Λ_i for $i = 1, 2$. Recall in particular that $\hat{W}_i = W_i \times \mathcal{M}_i$ acts faithfully on Λ_i .
- Let $\varphi : \Lambda_1 \rightarrow \Lambda_2$ be a ring homomorphism and $\psi : \hat{W}_1 \rightarrow \hat{W}_2$ be a group homomorphism such that

$$\varphi(w(a)) = \psi(w)(\varphi(a)), \quad \forall a \in \Lambda_1, \forall w \in \hat{W}_1. \quad (3.1)$$

Example 7

$(\mathbb{C}[x_1, x_2, \dots, x_n], A_n, 1)$ and $(\mathbb{C}[x_1, x_2, \dots, x_n], S_n, 1)$ with φ as the identity map, and $\psi : A_n \rightarrow S_n$ the standard inclusion.

Theorem 8 (J*)

- *There is an algebra homomorphism*

$$\Phi : \text{Frac}(\Lambda_1) \# \hat{W}_1 \rightarrow \text{Frac}(\Lambda_2) \# \hat{W}_2 \quad (3.2)$$

given by

$$\Phi(fw) = \varphi(f)\psi(w), \quad f \in \text{Frac}(\Lambda_1), w \in \hat{W}_1 \quad (3.3)$$

- *Suppose there is a subspace U of Λ_2 such that $\Lambda_2 \cong \varphi(\Lambda_1) \otimes U$ as $\psi(\hat{W}_1)$ -modules, where $\psi(\hat{W}_1)$ acts on $\varphi(a) \otimes u$ by*

$$\psi(w)(\varphi(a) \otimes u) = \psi(w)(\varphi(a)) \otimes u = \varphi(w(a)) \otimes u$$

then Φ restricts to an algebra homomorphism

$$\Phi : \mathcal{F}_{\Lambda_1} \rightarrow \mathcal{F}_{\Lambda_2} \quad (3.4)$$

Theorem 9 (J^*)

Let $(\Lambda_1, W_1, \mathcal{M}_1)$ and $(\Lambda_2, W_2, \mathcal{M}_2)$ be flag order data and $\mathcal{F}_{\Lambda_1}, \mathcal{F}_{\Lambda_2}$ be the corresponding standard flag orders such that the following are true:

- $\Lambda_2 = \Lambda_1 \oplus I$, where $I = (a_1, a_2, \dots, a_n)$ is an ideal of Λ_2 ,
- there is an embedding $\hat{W}_1 \rightarrow \hat{W}_2$ that satisfies the Condition 3.1 with the natural embedding of $\Lambda_1 \rightarrow \Lambda_2$,
- for every $w \in \hat{W}_1$ and $a_i \in I = (a_1, a_2, \dots, a_n)$, $w(a_i) = a_i$.

Then $\mathcal{F}_{\Lambda_2} \cap (\text{Frac}(\Lambda_1) \# \hat{W}_1) = \mathcal{F}_{\Lambda_1}$. In particular, $\mathcal{F}_{\Lambda_1} \hookrightarrow \mathcal{F}_{\Lambda_2}$.

Example 10

$(\mathbb{C}[x_1, x_2, \dots, x_n], S_n, 1)$, $(\mathbb{C}[x_1, x_2, \dots, x_n], A_n, 1)$, and $I = \langle 0 \rangle$

Definition 11

Given an ideal $I \subset \Lambda$, we define:

$$\mathcal{F}_\Lambda[I] = \{X \in \mathcal{F}_\Lambda \mid X(I) \subset I\}$$

the *subring of \mathcal{F}_Λ that fixes I* .

Definition 12

Given an ideal $I \subset \Lambda$, we define:

$$I\mathcal{F}_\Lambda = \{X \in \mathcal{F}_\Lambda \mid X(\Lambda) \subset I\}$$

the *subring of \mathcal{F}_Λ send Λ to I* . In fact, $I\mathcal{F}_\Lambda$ is an ideal of $\mathcal{F}_\Lambda[I]$.

Theorem 13 (J*)

Following the same assumptions as in Theorem 9, we have an embedding

$$\eta: \mathcal{F}_{\Lambda_1} \hookrightarrow \mathcal{F}_{\Lambda_2}[I]/I\mathcal{F}_{\Lambda_2}$$

- This is quite similar to the situation of differential operators acting on a polynomial ring over an algebraically closed field and a quotient of that ring.
- In that situation the map is surjective as well, but η is not always surjective.

- Let $(\Lambda_1, W_1, \mathcal{M}_1)$, $(\Lambda_2, W_2, \mathcal{M}_2)$ be two flag order data, let each Λ_i be a \mathbb{k} -algebra, and $\mathcal{F}_i = \text{Frac}(\Lambda_i) \# \hat{W}_i$. Put $\otimes = \otimes_{\mathbb{k}}$.
- Let $\Lambda = \Lambda_1 \otimes \Lambda_2$. Then $(\Lambda, W_1 \times W_2, \mathcal{M}_1 \times \mathcal{M}_2)$ is a flag order datum with standard flag order \mathcal{F}_Λ .
- Let $(\mathcal{F}_1 \otimes \mathcal{F}_2)_\Lambda = \{X \in \mathcal{F}_1 \otimes \mathcal{F}_2 \mid X(\Lambda) \subset \Lambda\}$.

Theorem 14 (J*)

There is a chain of embeddings

$$\mathcal{F}_{\Lambda_1} \otimes \mathcal{F}_{\Lambda_2} \hookrightarrow \mathcal{F}_\Lambda \hookrightarrow (\mathcal{F}_1 \otimes \mathcal{F}_2)_\Lambda.$$

Corollary 15 (J*)

The tensor product of two principal flag orders is a principal flag order.

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Thank you. Questions?