On related standard flag orders

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AMS Spring Central Sectional Meeting

March 27th, 2022

1. Preliminary Materials
   - History and Motivation
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History and Motivation

- These types of objects are a generalization of the framework of *Harish-Chandra modules* where $\mathscr{U}$ is the universal enveloping algebra of a reductive Lie algebra and Γ is the universal enveloping algebra of a Cartan subalgebra (generalized weight modules) [DFO94].
- The *Galois rings* and *Galois orders* were originally defined and studied by Futorny and Ovsienko in 2010.
- They form a collection of algebras that contains many important examples:
  - *Generalized Weyl algebras* defined by independently by Bavula and Rosenberg in the early nineties.
  - Universal enveloping algebra of $\mathfrak{gl}_n$
  - Finite W-algebras

Galois Rings and Galois Orders

- Let Λ be a Noetherian integrally closed domain, G a finite subgroup of $\operatorname{Aut}\Lambda$, and $\mathscr{M}$ a submonoid of $\operatorname{Aut}\Lambda$, normalized by G, such that $\mathscr{M}\mathscr{M}^{-1} \cap G = 1$. The triple $(\Lambda, G, \mathscr{M})$ is called a *Galois order datum*.
- Let $\Gamma := \Lambda^G$

Definition 1

Given a commutative ring R and a submonoid $\mathscr{M} \subseteq \operatorname{Aut} R$, we define the *smash product* as follows:

$$R\#\mathscr{M} := \{\sum_{\mu\in\mathscr{M}} a_\mu \mu \mid a_\mu \in R \text{ and finitely many } a_\mu \neq 0\},$$

with component-wise addition, and multiplication defined by $a_1\mu_1 \cdot a_2\mu_2 = (a_1\mu_1(a_2))\mu_1\mu_2$ and extending bilinearly.

- Since G acts on Λ, its action naturally extends to an action on $\operatorname{Frac}(\Lambda)$.
- As such, G acts on $\operatorname{Frac}(\Lambda)\#\mathscr{M}$: $g(a\mu) = g(a) \cdot {}^{g}\mu$, where ${}^{g}\mu = g\mu g^{-1}$
- We have the following commutative diagram:

$$\begin{array}{ccccc} \Lambda & \hookrightarrow & \operatorname{Frac}(\Lambda) & \hookrightarrow & \operatorname{Frac}(\Lambda)\#\mathscr{M} \\ \uparrow & & \uparrow & & \uparrow \\ \Gamma & \hookrightarrow & \operatorname{Frac}(\Gamma) & \hookrightarrow & (\operatorname{Frac}(\Lambda)\#\mathscr{M})^{G} \end{array}$$

- Note: $\operatorname{Frac}(\Lambda)/\operatorname{Frac}(\Gamma)$ is a Galois extension with Galois group G.

Example 2

$(\mathbb{C}[x_1, x_2], S_2, 1)$ is a Galois order datum.

- Galois Γ-rings are particular subalgebras of $(\mathrm{Frac}(\Lambda)\#\mathscr{M})^G$ containing Γ.
- In [Har20], Hartwig showed the following

Theorem 3 (Hartwig)

Let $\mathscr{U}$ be a Galois Γ-ring such that $X(\Gamma) \subseteq \Gamma$ for every $X \in \mathscr{U}$. Then $\mathscr{U}$ is a Galois Γ-order.

- A Galois Γ-ring satisfying this condition is called a *principal Galois Γ-order*.
- Note: Γ is maximal commutative in any Galois Γ-order.

Example 4

$(\Lambda\#\mathscr{M})^G$ is a (principal) Galois Γ-order.

- They help us to study Gelfand-Tsetlin modules.

Definition 5

A $\mathscr{U}$-module V is a *Gelfand-Tsetlin* module (with respect to Γ) if $\dim(\Gamma.v) < \infty$ for all $v \in V$.

Example 6

Any finite-dimensional weight module is a Gelfand-Tsetlin module.

The major results in [FO14] give:

1. The existence of "generic" simple Gelfand-Tsetlin modules over Galois rings.
2. A "rough" classification of simple Gelfand-Tsetlin modules over Galois orders.

Flag Orders

- In [Web19], Webster introduced principle flag orders.
- The *flag order datum* is a triple $(\Lambda, W, \mathscr{M})$ under the same restrictions as before (Replace G with W from the Galois order datum).
- Instead of looking at invariants however, we look at $\mathrm{Frac}(\Lambda)\#(W \ltimes \mathscr{M})$.
- Sometimes $W \ltimes \mathscr{M}$ is denoted by $\hat{W}$ for shorter notation.

Definition 7

A *principal flag order* with data $(\Lambda, W, \mathscr{M})$ is a subalgebra $F \subset \mathrm{Frac}(\Lambda)\#(W \ltimes \mathscr{M})$ such that:

(i) $\Lambda\#W \subset F$,

(ii) $\mathrm{Frac}(\Lambda)F = \mathrm{Frac}(\Lambda)\#(W \ltimes \mathscr{M})$,

(iii) For every $X \in F$, $X(\Lambda) \subset \Lambda$.

Definition 8

The *standard flag order* with data $(\Lambda, W, \mathscr{M})$ is the subalgebra of all elements X in $\mathrm{Frac}(\Lambda)\#(W \ltimes \mathscr{M})$ satisfying (iii) and is denoted $\mathcal{F}_\Lambda$.

Example 9

If $\Lambda = \mathbb{C}[x_1, x_2, \ldots, x_n]$ and $\hat{W}$ is a finite complex reflection group acting on $\mathbb{C}^n$, then $\mathcal{F}_\Lambda$ is the nilHecke algebra of $\hat{W}$ (see [Web19]).

- Let $(\Lambda_1, W_1, \mathscr{M}_1)$, $(\Lambda_2, W_2, \mathscr{M}_2)$ be two flag order data, and L_i the field of fractions of Λ_i for $i = 1, 2$. Recall in particular that $\hat{W}_i = W_i \ltimes \mathscr{M}_i$ acts faithfully on Λ_i.
- Let $\varphi : \Lambda_1 \to \Lambda_2$ be a ring homomorphism and $\psi : \hat{W}_1 \to \hat{W}_2$ be a group homomorphism such that

$$\varphi\big(w(a)\big) = \psi(w)\big(\varphi(a)\big), \qquad \forall a \in \Lambda_1, \forall w \in \hat{W}_1. \tag{3.1}$$

Example 10

$(\mathbb{C}[x_1, x_2, \ldots, x_n], A_n, 1)$ and $(\mathbb{C}[x_1, x_2, \ldots, x_n], S_n, 1)$ with φ as the identity map, and $\psi\colon A_n \to S_n$ the standard inclusion.

Theorem 11 (Hartwig, J*)

- *There is an algebra homomorphism*

$$\Phi : L_1 \# \hat{W}_1 \to L_2 \# \hat{W}_2 \tag{3.2}$$

given by

$$\Phi(fw) = \varphi(f)\psi(w), \qquad f \in L_1, w \in \hat{W}_1 \tag{3.3}$$

- *Suppose there is a subspace* U *of* Λ_2 *such that* $\Lambda_2 \cong \varphi(\Lambda_1) \otimes U$ *as* $\psi(\hat{W}_1)$*-modules, where* $\psi(\hat{W}_1)$ *acts on* $\varphi(a) \otimes u$ *by*

$$\psi(w)\big(\varphi(a) \otimes u\big) = \psi(w)\big(\varphi(a)\big) \otimes u = \varphi(w(a)) \otimes u$$

then Φ *restricts to an algebra homomorphism*

$$\Phi : (L_1 \# \hat{W}_1)_{\Lambda_1} \to (L_2 \# \hat{W}_2)_{\Lambda_2} \tag{3.4}$$

Theorem 12 (J*)

Let $(\Lambda, W, \mathscr{M})$ *and* $(\Lambda', W', \mathscr{M}')$ *be flag order data and* $\mathcal{F}_\Lambda, \mathcal{F}'_{\Lambda'}$ *be the corresponding standard flag orders such that the following are true:*

- $\Lambda = \Lambda' \oplus I$, *where* I *is an ideal of* Λ,
- *there is an embedding* $\hat{W}' \to \hat{W}$ *that satisfies the Condition 3.1 with the natural embedding of* $\Lambda' \to \Lambda$,
- *for every* $w \in \hat{W}'$ *and* $a \in I$, $w(a) = a$.

Then $\mathcal{F}_\Lambda \cap (\mathrm{Frac}(\Lambda')\#\hat{W}') = \mathcal{F}'_{\Lambda'}$. *In particular,* $\mathcal{F}'_{\Lambda'} \hookrightarrow \mathcal{F}_\Lambda$.

Example 13

$(\mathbb{C}[x_1, x_2, \ldots, x_n], S_n, 1)$, $(\mathbb{C}[x_1, x_2, \ldots, x_n], A_n, 1)$, and $I = \langle 0 \rangle$

Definition 14

Given an ideal $I \subset \Lambda$, we define:

$$\mathcal{F}_\Lambda[I] = \{X \in \mathcal{F}_\Lambda \mid X(I) \subset I\}$$

the *subring of $\mathcal{F}_\Lambda$ that fixes I*.

Definition 15

Given and ideal $I \subset \Lambda$, we define:

$$I\mathcal{F}_\Lambda = \{X \in \mathcal{F}_\Lambda \mid X(\Lambda) \subset I\}$$

the *subring of $\mathcal{F}_\Lambda$ send Λ to I.* In fact, $I\mathcal{F}_\Lambda$ is an ideal of $\mathcal{F}_\Lambda[I]$.

Lemma 16

The map $\mathcal{F}_\Lambda[I]/I\mathcal{F}_\Lambda \to \operatorname{End}\Lambda'$ *is injective.*

Proof.

First we observe that $\mathcal{F}_\Lambda[I] \to \operatorname{End}(\Lambda')$ by sending $X \mapsto (a + I \mapsto X(a) + I)$. We now claim the kernel of this map is $K = I\mathcal{F}_\Lambda$. It is clear that $K \supset I\mathcal{F}_\Lambda$, and if $X \in K$ then $X(a+I) = I$, that is $X(a) \in I$ for all $a \in \Lambda$. It follows that $X \in I\mathcal{F}_\Lambda$. Hence, the map is injective. □

Theorem 17 (J*)

Following the same assumptions as in Theorem 12, we have an embedding $\eta\colon \mathcal{F}'_{\Lambda'} \hookrightarrow \mathcal{F}_{\Lambda}[I]/I\mathcal{F}_{\Lambda}$

- This is quite similar to the situation of differential operators acting on a polynomial ring over an algebraically closed field and a quotient of that ring.
- In that situation the map is surjective as well, but η is not always surjective.

- Let $(\Lambda_1, W_1, \mathscr{M}_1)$, $(\Lambda_2, W_2, \mathscr{M}_2)$ be two flag order data, let each Λ_i be a $\mathbb{k}$-algebra, and $\mathcal{F}_i = \mathrm{Frac}(\Lambda_i)\#\hat{W}_i$. Put $\otimes = \otimes_{\mathbb{k}}$.
- Let $\Lambda = \Lambda_1 \otimes \Lambda_2$. Then $(\Lambda, W_1 \times W_2, \mathscr{M}_1 \times \mathscr{M}_2)$ is a flag order datum with standard flag order $\mathcal{F}_\Lambda$.
- Let $(\mathcal{F}_1 \otimes \mathcal{F}_2)_\Lambda = \{X \in \mathcal{F}_1 \otimes \mathcal{F}_2 \mid X(\Lambda) \subset \Lambda\}$.

Theorem 18 (J*)

There is a chain of embeddings

$$\mathcal{F}_{\Lambda_1} \otimes \mathcal{F}_{\Lambda_2} \hookrightarrow \mathcal{F}_\Lambda \hookrightarrow (\mathcal{F}_1 \otimes \mathcal{F}_2)_\Lambda.$$

Corollary 19 (J*)

The tensor product of two principal flag orders is a principal flag order.

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Thank you. Questions?